A Model of Product Awareness and Industry Life Cycles
Online Technical Appendix

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All references to the Main paper are prefixed by "Main". Define the partial derivative with respect to a as the operator $\boldsymbol{\partial}_a$.

A Proofs with Differentiated Firms

This section nests the cases of symmetric and asymmetric firms: Appendix D and Main Section 2. Here, the additional quality term q_{im} is isomorphic(for all the usual reasons) to adding an idiosyncratic productivity for each firm (i, m).

A.1 State Space of Consumers and Information Sets

This section provides more details on tracking awareness with differentiated firms. For the most generality, I use the joint distribution of all consumer states (i.e., awareness sets A_j and idiosyncratic preferences ξ_j) to construct the marginal pmf $\hat{f}(a, A)$ over all the possible awareness set types. This is required because, in general, the count pmf, $f_n(a)$, is no longer sufficient for firm and consumer decisions with full differentiation.

Distribution of Consumer States Recall that consumer j is heterogeneous over both choice sets $A_{jm}(a)$ and permanent idiosyncratic match preferences ξ_{mj} for all industries. To forecast profits and make optimal pricing decisions, firms need to form expectations over the evolution of this state space. I will maintain assumptions to ensure that these states are independent across industries.

Define the joint distribution of consumer states for a particular industry as $\hat{\Psi}(a, A_j(a), \xi_j)$, with density pdf $\hat{\psi}(a, \cdot)$. This must sum to 1 for all consumers at each industry age a.

To aid in computation, factor this into two marginals: (1) the *marginal* probability mass function (pmf) of the awareness states A across consumers as a discrete distribution over the power-set of $\mathcal{I}$, $\hat{f}(a, A) : \mathbf{2}^{\mathcal{I}} \to \mathbb{R}$; and (2) the marginal distribution of idiosyncratic preferences, $G(\xi) : \mathbb{R}^N \to \mathbb{R}$, with density $g(\xi)$. While we can leave these marginals fairly general, to simplify integrals, I will maintain the following assumption throughout this paper:[1]

Assumption 1 (Independence of Preferences and Awareness)**.** *Assume conditions such that: (1) awareness evolves independent of preferences—i.e., $\hat{\psi}(a, A, \xi) = \hat{f}(a, A)g(\xi)$— and the distribution of preferences is independent across industries: $\xi_{mj} \perp \xi_{m'j}$ for all $m \neq m'$; and (2) g is continuous.*

[1] An example of a model extension that breaks this assumption is if the evolution of $A_{jm}(a)$ is a function of the particular ξ_i for $i \in A_{jm}(a)$. An example of this is when a consumer controls the evolution of choice sets through shopping intensity lowers her intensity when she finds a high match value. These sorts of additions would not destroy the mechanism but would require more-complicated integrals for (A.3) and (A.30).

The canonical case of the idiosyncratic distribution of preferences is the independent product: $\xi \sim G \equiv G^u(\xi_1) \times \ldots G^u(\xi_N)$, where G^u is the univariate Gumbel distribution. In that case, the idiosyncratic preferences are independent across product categories, products, and consumers.

Information Sets and Timing To summarize the information structure and timing with this state-space: (1) consumers have incomplete information about firms, as captured in $A_j(a)$ choice sets, but complete information on their idiosyncratic preference ξ_i for $i \in A_j$; (2) firms have complete information on the joint distribution $\hat{\Psi}(\cdot)$ but cannot price discriminate based on awareness or preferences; (3) firms have complete information on the pricing and production decisions of other firms in the industry; (4) simultaneously, firms post prices, $p_i(a)$ (i.e., repeated Bertrand competition with no price discrimination), while consumers choose quantity demanded, $y_{ij}(a)$, to clear markets for each good; and (5) $A_j(a)$ stochastically evolves for each consumer as the industry ages (through any process, as long as Assumption 1 is maintained).

A.2 Consumers' Static Problem

In order to save on notation, the consumer's static problem written here uses y_{imj} directly instead of c_{imj}. As in models of monopolistic competition, this distinction is irrelevant if the aggregation technology (and/or preferences under the laws of large numbers of consumers, as proven in Appendix B.3) for creating consumer goods is identical to that for creating investment goods. To have a separate technology and productivity for the creation of capital and R&D goods, I would need to specify a different version of (A.1) for each—or simply undo the awareness-specific elements, as discussed in Section 3.2.

Suppress the m, t, and a indices where convenient. Assume that firms are differentiated, and may have a persistent quality difference, $q_{im} > 0$. Firm differentiation leads to an additional quality term in the objective compared to the static decisions of Main (3),

$$\max_{y_{imj} \geq 0} \left(\int_0^M \left(\sum_{i \in A_{mj}} q_{im} e^{\sigma \xi_{imj}} y_{imj} \right)^{\varsigma} \mathrm{d}m \right)^{1/\varsigma} \text{ s.t. } \int_0^M \left[\sum_{i \in A_{mj}} \hat{p}_{im} y_{imj} \right] \mathrm{d}m \leq P_j \Omega_j \tag{A.1}$$

Assumption 2 (Degree of Differentiation)**.** *Assume that* $0 < \sigma < \frac{1}{\kappa - 1}$ *and* $\xi_j \sim$ *Gumbel.*

Define the set of industries with awareness of at least one firm as $\mathcal{M}_j \equiv \{m \in [0, M] \text{ s.t. } |\mathcal{I}_{jm}| > 0\}$.

Definition 1 (Total Demand)**.** *Given* $\hat{\Psi}(\cdot)$*, the total demand for firm* i *is defined as,*

$$y_i(a, \vec{p}) \equiv \int y_{ij}(\vec{p}, \xi_{ij}) \mathbb{1}\left\{\textit{Choose } i \textit{ from } A_j \textit{ given } \vec{p} \textit{ and } \xi_j\right\} \mathrm{d}\hat{\Psi}(a, A_j, \xi_j) \tag{A.2}$$

Proposition 1 (Intensive Demand with Differentiated Firms)**.** *Given real prices* p *and real income* Ω*, for each industry* m*:*

1. *Of those with non-empty awareness sets, almost every consumer purchases from a single firm per industry. A consumer purchases product* i *and no others if and only if*

$$\log\left(\frac{p_{i'}}{q_{i'}}\right) - \log\left(\frac{p_i}{q_i}\right) > \sigma\left(\xi_{i'j} - \xi_{ij}\right), \quad \forall\, i' \in A_j \setminus \{i\} \tag{A.3}$$

2. *The intensive demand for product* i *is*

$$y_{ij}(a, \xi_{ij}) = q_i^{\kappa-1} e^{\sigma(\kappa-1)\xi_{ij}} p_i^{-\kappa} \Omega_j, \quad y_{i'j} = 0, \forall\, i' \in A_j \setminus \{i\} \tag{A.4}$$

3. *The price index is a function of the preferences, ξ_{mj}, firm average quality q_m, and nominal prices, $\hat{p}_m$,*

$$P_j \equiv \left(\int_{\mathcal{M}_j} \left(e^{\sigma \xi_{imj}} q_{im} \right)^{\kappa-1} \hat{p}_{im}^{1-\kappa} \mathrm{d}m \right)^{\frac{1}{1-\kappa}} \tag{A.5}$$

Proof of Proposition 1. First, define the Lagrangian for the consumers' optimization problem in (A.1) with $\lambda_j > 0$ as the Lagrange multiplier on the budget constraint, and $\mu_{imj} \geq 0$ as the Lagrange multipliers ensuring weak positivity on every choice of y_{imj}.

$$\mathcal{L} = \left(\int_0^M \left(\sum_{i' \in A_m} q_{i'm} e^{\sigma \xi_{i'mj}} y_{i'mj} \right)^{\varsigma} \mathrm{d}m \right)^{1/\varsigma} - \lambda_j \left(\int_0^M \sum_{i' \in A_m} \hat{p}_{i'm} y_{i'mj} \mathrm{d}m - P\Omega \right) + \int_0^M \sum_{i' \in A_m} \mu_{i'mj} y_{i'mj} \mathrm{d}m \tag{A.6}$$

The first-order necessary conditions with respect to y_{imj} are

$$S_j \left(\sum_{i' \in A_m} e^{\sigma \xi_{i'mj}} q_{i'm} y_{i'mj} \right)^{\varsigma - 1} q_{im} e^{\sigma \xi_{imj}} = \lambda_j \hat{p}_{im} - \mu_{imj} \tag{A.7}$$

$$\lambda_j > 0, \quad \mu_{imj} \geq 0, \quad \mu_{imj} y_{imj} = 0, \quad \forall\, i, m \tag{A.8}$$

with the following definition:

$$S_j \equiv \left(\int_0^M \left(\sum_{i' \in A_m} q_{i'm} e^{\sigma \xi_{i'mj}} y_{i'mj} \right)^{\varsigma} \mathrm{d}m \right)^{1/\varsigma - 1} \tag{A.9}$$

Intensive Demand: Maintain Assumption 2 throughout. Assume, to be verified, that if $|A_m| > 0$, the consumer will almost certainly consume a single product per industry.

To solve for λ_j and the price index, I will follow standard CES algebra under the assumption of consuming, at most, one product per industry. The i index is dropped since there is only one good per industry, and the j index is dropped for simplicity—though given the assumptions for the aggregate consumer, I show in Appendix B.3 that S_j is identical for all j, and, hence, all consumers have the same price index. From (A.7), for industries with non-empty awareness sets,

$$S_j \left(e^{\sigma \xi_m} q_m \right)^{\varsigma} y_m^{\varsigma - 1} = \lambda_j \hat{p}_m \tag{A.10}$$

Take the ratio of two industries m' and m with positive demand

$$\frac{y_{m'}^{\varsigma-1}}{y_m^{\varsigma-1}} = \left(\frac{e^{\sigma \xi_{m'}} q_{m'}}{e^{\sigma \xi_m} q_m} \right)^{-\varsigma} \frac{\hat{p}_{m'}}{\hat{p}_m} \tag{A.11}$$

Rearrange,

$$y_{m'} = y_m \hat{p}_m^{\kappa} \left(e^{\sigma \xi_m} q_m \right)^{1-\kappa} \hat{p}_{m'}^{-\kappa} \left(e^{\sigma \xi_{m'}} q_{m'} \right)^{\kappa - 1} \tag{A.12}$$

Multiply both sides by $\hat{p}_{m'}$ and integrate over all industries with positive consumption, $m' \in \mathcal{M}_j$,

$$\int_{\mathcal{M}_j} \hat{p}_{m'} y_{m'} \mathrm{d}m' = y_m \hat{p}_m^{\kappa} \left(e^{\sigma \xi_m} q_m \right)^{1-\kappa} \int_{\mathcal{M}_j} \left(e^{\sigma \xi_{m'}} q_{m'} \right)^{\kappa-1} \hat{p}_{m'}^{1-\kappa} \mathrm{d}m' \tag{A.13}$$

Recognize that industry m is infinitesimal, so the integrals are identical with or without industry m. Hence, the consumer cannot affect the price index either through a change in the intensive demand or by switching between different i in $\mathcal{I}_m$. Define the price index as

$$P_j \equiv \left(\int_{\mathcal{M}_j} \left(e^{\sigma \xi_m} q_m \right)^{\kappa-1} \hat{p}_m^{1-\kappa} \mathrm{d}m \right)^{\frac{1}{1-\kappa}} \tag{A.14}$$

Reorganize (A.13) using the price index, noting that the left-hand side of the equality is the budget

$$y_m = \left(e^{\sigma \xi_m} q_m \right)^{\kappa-1} \left(\frac{\hat{p}_m}{P} \right)^{-\kappa} \Omega \tag{A.15}$$

This function is the intensive demand, as in Main (6) and (A.4). Note that since $\kappa > 1$, $\frac{\partial y_i}{\partial \hat{p}_i} < 0$, $\frac{\partial y_i}{\partial q_i} > 0$, $\frac{\partial y_i}{\partial \xi_i} > 0$. Also, if the real income and nominal price, $\hat{p}$, are is kept constant, then $\frac{\partial y_i}{\partial P} > 0$, reflecting substitution away from other goods to this industry.

Extensive Demand: The proof strategy is to assume that a single product is consumed, to use the non-negativity of the Lagrange multipliers for the other products to determine a set of inequalities necessary for this choice to hold, and then to show that multiple products will be chosen only under measure 0 events. The inequality constraints in (A.7) for all products $i, i' \in \mathcal{I}_m$ give

$$S_j \left(e^{\sigma \xi_i} q_i y_i \right)^{\varsigma-1} q_{i'} e^{\sigma \xi_{i'}} \leq \lambda_j \hat{p}_{i'} \tag{A.16}$$

Rearrange (A.10)

$$\left(y_i q_i e^{\sigma \xi_i} \right)^{\varsigma-1} = \frac{\lambda_j}{S_j} \frac{\hat{p}_i}{q_i e^{\sigma \xi_i}} \tag{A.17}$$

Combine with (A.16)

$$\frac{\lambda_j}{S_j} \frac{\hat{p}_i}{q_i} e^{-\sigma \xi_{ij}} q_{i'} e^{\sigma \xi_{i'j}} \leq \frac{\lambda_j}{S_j} \hat{p}_{i'} \tag{A.18}$$

Take logs and rearrange

$$\log \left(\frac{\hat{p}_{i'}}{q_{i'}} \right) - \log \left(\frac{\hat{p}_i}{q_i} \right) \geq \sigma (\xi_{i'j} - \xi_{ij}) \tag{A.19}$$

Using $p \equiv \hat{p}/P$, this expression gives (A.3). Finally, show that only measure 0 consumers choose multiple products. Without loss of generality, assume that there are only two products in the industry and that $y_1 > 0$ and $y_2 > 0$. The first-order conditions are then

$$S_j \left(e^{\sigma \xi_{ij}} q_i y_{ij} + e^{\sigma \xi_{i'j}} q_{i'} y_{i'j} \right)^{\varsigma-1} q_i e^{\sigma \xi_{ij}} = \lambda_j \hat{p}_i \tag{A.20}$$

$$S_j \left(e^{\sigma \xi_{ij}} q_i y_{ij} + e^{\sigma \xi_{i'j}} q_{i'} y_{i'j} \right)^{\varsigma-1} q_{i'} e^{\sigma \xi_{i'j}} = \lambda_j \hat{p}_{i'} \tag{A.21}$$

Take the ratio and the log to find an equation in ξ space,

$$\xi_{i'j} - \xi_{ij} = \sigma^{-1} \left(\log \left(\frac{\hat{p}_{i'}}{q_{i'}} \right) - \log \left(\frac{\hat{p}_i}{q_i} \right) \right) \tag{A.22}$$

For a given set of prices and distribution of ξ, there are an infinite number of agents with the particular combination of these $(\xi_{i'j}, \xi_{ij})$. However, the solution is an affine subset of the ξ_1, ξ_2 space. Given the independence of the ξ preferences by assumption, the measure of this affine subset is 0. The conclusion is that the set of agents who purchase multiple products is measure 0 if prices are positive, and (A.19) can be written as a strict inequality for almost every consumer. $\square$

A.3 Total Demand

In this section, I assume conditions such that consumers have identical real incomes, Ω, and I derive the demand curve faced by a firm. Due to the intensive margin, the market shares for firms are less useful than in a discrete-choice model. I assume that the consumers have identical nominal incomes, and that they have the same price index in (A.5) due to a law of large numbers—as proven in Appendix B.

Asymmetric Match Quality To nest the versions of the model used in Appendix B.8, this derivation will consider cases where the match distributions for firm i may be different from those of all other firms (which are independent and identical to each other). In particular, for some μ_i and $\mu_{\text{-}i}$, the CDF of the matches follows a Gumbel distribution centered at $\log \mu_i$,

$$G_i(x) = e^{-e^{-x+\log \mu_i}} = e^{-\mu_i e^{-x}} \tag{A.23}$$

And for all other firms in the awareness set, $i' \neq i$ follow indpendent distributions centered at $\log \mu_{\text{-}i}$,

$$G_{i'}(x) = e^{-e^{-x+\log \mu_{\text{-}i}}} \tag{A.24}$$

$$= \underbrace{e^{-e^{-x}}}_{\text{Symmetric}} e^{-(\mu_{\text{-}i}-1)e^{-x}} \tag{A.25}$$

This recenters the distribution by $\log \mu$ for firm i, and by $\log \mu_{\text{-}i}$ for the other firms. Presumably, $\mu_{\text{-}i} \leq \mu$ so that matches are better for a particular firm. Note that if $\mu = \mu_{\text{-}i} = 1$ there is no match asymmetry and the model has IID gumbel draws. Calculating the PDF,

$$dG_i(x) = \mu e^{-x-\mu_i e^{-x}} \mathrm{d}x \tag{A.26}$$

$$= \underbrace{e^{-x-e^{-x}}}_{\text{Symmetric}} \mu e^{-(\mu-1)e^{-x}} \mathrm{d}x \tag{A.27}$$

$$dG_{i'}(x) = e^{-x-e^{-x}} \mu_{\text{-}i} e^{-(\mu_{\text{-}i}-1)e^{-x}} \mathrm{d}x \tag{A.28}$$

Recall that

$$\bar{\Gamma} \equiv \Gamma(1-\sigma(\kappa-1))^{1/(1-\kappa)} \tag{A.29}$$

Define the set of possible awareness states that contain product i as $A \,|\, i \equiv \left\{A \,|A \in \mathbf{2}^{\mathcal{I}} \text{ s.t. } i \in A\right\}$.

Proposition 2 (Total Demand for Gumbel Preferences)**.** *Given variations on the symmetry of q_i and μ_i,*

1. *Asymmetric quality and symmetric matches. i.e. $\mu = \mu_{\text{-}i} = 1$*

$$y_i(a, \vec{p}) = \bar{\Gamma}^{1-\kappa} \Omega \, q_i^{\kappa-1} p_i^{-\kappa} \sum_{A\,|\,i} \left(\hat{f}(a, A) \left[\sum_{i' \in A} \left(\frac{q_i}{q_{i'}} \frac{p_{i'}}{p_i} \right)^{\text{-}1/\sigma} \right]^{\sigma(\kappa-1)-1} \right) \tag{A.30}$$

2. *Symmetric quality and asymmetric matches. i.e. $q = \bar{\Gamma}$*

$$y_i(a, \vec{p}) = \mu_i p_i^{-\kappa} \Omega \sum_{A\,|\,i} \left[\hat{f}(a, A) \left(\mu_i + \mu_{\text{-}i} \sum_{i' \in A \setminus i} \left(\frac{p_{i'}}{p_i} \right)^{\text{-}1/\sigma} \right)^{\sigma(\kappa-1)-1} \right] \tag{A.31}$$

3. Symmetric quality and symmetric matches. i.e. $q = \bar{\Gamma}$ *and* $\mu = \mu_{\text{-}i} = 1$

$$y_i(a, \vec{p}) = \Omega\, p_i^{-\kappa} \sum_{A\,|\,i} \left(\hat{f}(a, A) \left[\sum_{i' \in A} \left(\frac{p_{i'}}{p_i} \right)^{\text{-}1/\sigma} \right]^{\sigma(\kappa-1)-1} \right) \tag{A.32}$$

4. Monopolistic competition. i.e. $N = 1$

$$y_i(a, p_i) = (1 - \hat{f}(a, \emptyset))\bar{\Gamma}^{1-\kappa} q_i^{\kappa-1} p_i^{-\kappa}\Omega \tag{A.33}$$

Proof. Since intensive demand is a function of ξ_j, to find the total demand for product i, the firm will sum up demand with ξ_{ij} conditional on product i being chosen. From (A.2), define the total demand for product i given a price vector p as

$$y_i(a, \vec{p}) = \sum_{A\,|\,i} \hat{f}(a, A) \int y_i(\xi_{ij}) \mathbb{1} \left\{ \log\left(\tfrac{p_{i'}}{q_{i'}}\right) - \log\left(\tfrac{p_i}{q_i}\right) > \sigma(\xi_{i'j} - \xi_{ij}) | \forall\, i' \in A \setminus i \right\} \mathrm{d}G(\xi_j) \tag{A.34}$$

Simplify using Assumption 1 and (A.4):

$$y_i(a, \vec{p}) = p_i^{-\kappa} q_i^{\kappa-1} \Omega \sum_{A\,|\,i} \hat{f}(a, A) \int e^{\sigma(\kappa-1)\xi_{ij}} \mathbb{1} \left\{ \log\left(\tfrac{p_{i'}}{q_{i'}}\right) - \log\left(\tfrac{p_i}{q_i}\right) > \sigma(\xi_{i'j} - \xi_{ij}) | \forall\, i' \in A \setminus i \right\} \mathrm{d}G(\xi_j) \tag{A.35}$$

This equation sums the demand across the distribution of A. For a particular A, find the total demand from agents conditional on having awareness set A

$$y_i(\vec{p}, A) \equiv p_i^{-\kappa} q_i^{\kappa-1} \Omega \int e^{\sigma(\kappa-1)\xi_{ij}} \mathbb{1} \left\{ \log\left(\tfrac{p_{i'}}{q_{i'}}\right)/\sigma - \log\left(\tfrac{p_i}{q_i}\right)/\sigma + \xi_{ij} > \xi_{i'j} | \forall\, i' \in A \setminus i \right\} \mathrm{d}G(\xi_j) \tag{A.36}$$

Define the marginal distribution of ξ_j for all products other than product i as $G_{-ij}(\xi_{-ij})$. For arbitrary $g(\xi)$, this expression could be calculated numerically. For iid Gumbel distributions with pdf $g(\xi_i)$, the integral is solved in two parts: first, use Fubini's Theorem and Assumption 1 to solve for the inner non-ξ_i variables, defined as ξ_{-ij}; and then integrate with respect to the the ξ_i variable. This is the standard technique in the derivation of the Logit probabilities.

$$\frac{y_i(\vec{p}, A)}{p_i^{-\kappa} q_i^{\kappa-1}\Omega} = \int_{-\infty}^{\infty} e^{\sigma(\kappa-1)\xi_{ij}} \left[\int \mathbb{1} \left\{ \log\left(\tfrac{p_{i'}}{q_{i'}}\right)/\sigma - \log\left(\tfrac{p_i}{q_i}\right)/\sigma + \xi_{ij} > \xi_{i'j} | \forall\, i' \in A \setminus i \right\} \mathrm{d}G_{\text{-}i}(\xi_{-ij}) \right] \mathrm{d}G_i(\xi_{ij}) \tag{A.37}$$

The inner integral is the cdf of the joint distribution of ξ_{-ij}. Recall the maintained assumption of the independence of the distribution of matches across types. Use the cdf of the Gumbel along each dimension other than i using (A.24)

$$\frac{y_i(\vec{p}, A)}{p_i^{-\kappa} q_i^{\kappa-1}\Omega} = \int_{-\infty}^{\infty} e^{\sigma(\kappa-1)\xi_{ij}} \prod_{i' \in A \setminus i} e^{-e^{-\left(\log(\frac{p_{i'}}{q_{i'}})/\sigma - \log(\frac{p_i}{q_i})/\sigma + \xi_{ij}\right) + \log \mu_{\text{-}i}}} \mathrm{d}G_i(\xi_{ij}) \tag{A.38}$$

Substitute in the pdf of the Gumbel with the distorted mean in (A.27)

$$= \int_{-\infty}^{\infty} e^{\sigma(\kappa-1)\xi_{ij}} \left(\prod_{i' \in A \setminus i} e^{-e^{-\left(\log(\frac{p_{i'}}{q_{i'}})/\sigma - \log(\frac{p_i}{q_i})/\sigma + \xi_{ij}\right) + \log \mu_{\text{-}i}}} \right) e^{-\xi_{ij}} e^{-e^{-\xi_{ij}}} \mu e^{-(\mu_i - 1)e^{-\xi_{ij}}} \mathrm{d}\xi_{ij} \tag{A.39}$$

Recognize that $\log\left(\frac{p_i}{q_i}\right)/\sigma - \log\left(\frac{p_i}{q_i}\right)/\sigma = 0$. Factoring, note that

$$e^{-e^{-\xi_{ij}}} = e^{-e^{-\left(\log(\frac{p_i}{q_i})/\sigma - \log(\frac{p_i}{q_i})/\sigma + \xi_{ij}\right)}} \tag{A.40}$$

$$= e^{-e^{-\left(\log(\frac{p_i}{q_i})/\sigma - \log(\frac{p_i}{q_i})/\sigma + \xi_{ij}\right)+\log\mu_{\text{-}i}}} e^{(\mu_{\text{-}i}-1)e^{-\xi_{ij}}} \tag{A.41}$$

This expression allows us to combine the product for all i', including i. Subsitute (A.41) into (A.39)

$$\frac{y_i(\vec{p},A)}{p_i^{-\kappa}q_i^{\kappa-1}\Omega} = \int_{-\infty}^{\infty} e^{\sigma(\kappa-1)\xi_{ij}} \exp\left(-\sum_{i'\in A} e^{-\left(\log\left(\frac{p_{i'}}{q_{i'}}\right)/\sigma - \log\left(\frac{p_i}{q_i}\right)/\sigma + \xi_{ij}\right)+\log\mu_{\text{-}i}}\right) e^{-\xi_{ij}}\mu_i e^{(\mu_{\text{-}i}-1)e^{-\xi_{ij}}} e^{-(\mu_i-1)e^{-\xi_{ij}}}\,\mathrm{d}\xi_{ij} \tag{A.42}$$

$$= \int_{-\infty}^{\infty} e^{\sigma(\kappa-1)\xi_{ij}} \exp\left(-\sum_{i'\in A} e^{-\left(\log\left(\frac{p_{i'}}{q_{i'}}\right)/\sigma - \log\left(\frac{p_i}{q_i}\right)/\sigma + \xi_{ij}\right)+\log\mu_{\text{-}i}}\right) e^{-\xi_{ij}}\mu_i e^{-(\mu_i-\mu_{\text{-}i})e^{-\xi_{ij}}}\,\mathrm{d}\xi_{ij} \tag{A.43}$$

Simplify by factoring the exponential

$$= \int_{-\infty}^{\infty} \mu_i \exp\left(-(1-\sigma(\kappa-1))\xi_{ij}\right) \exp\left(-e^{-\xi_{ij}}\left(\mu_i - \mu_{\text{-}i} + \mu_{\text{-}i}\left(\frac{p_i}{q_i}\right)^{1/\sigma}\sum_{i'\in A}\left(\frac{p_{i'}}{q_{i'}}\right)^{\text{-}1/\sigma}\right)\right)\mathrm{d}\xi_{ij} \tag{A.44}$$

If $B > 0$ and $A > 0$, then $\int_{-\infty}^{\infty} e^{-Ax}e^{-Be^{-x}}\mathrm{d}x = B^{-A}\Gamma(A)$, where $\Gamma(\cdot)$ is the Gamma function.

$$= \mu_i\Gamma(1-\sigma(\kappa-1))\left(\mu_i - \mu_{\text{-}i} + \mu_{\text{-}i}\left(\frac{p_i}{q_i}\right)^{1/\sigma}\sum_{i'\in A}\left(\frac{p_{i'}}{q_{i'}}\right)^{\text{-}1/\sigma}\right)^{\sigma(\kappa-1)-1} \tag{A.45}$$

Where in the fully symmetric case with $\mu_i = \mu_{\text{-}i} = 1$,

$$= \Gamma(1-\sigma(\kappa-1))\left(\frac{p_i}{q_i}\right)^{\frac{\sigma(\kappa-1)-1}{\sigma}}\left(\sum_{i'\in A}\left(\frac{p_{i'}}{q_{i'}}\right)^{\text{-}1/\sigma}\right)^{\sigma(\kappa-1)-1} \tag{A.46}$$

Assumption 2 ensures that the variance of the idiosyncratic preferences is not so large that the total demand explodes as the demand from agents with large ξ_{ij} is summed. To find the total demand for product i given price vectors p, integrate over the distribution of A states in the economy. From (A.7) and (A.35), the total demand is the sum of all awareness states that contain product i in (A.35), where we can substitute using (A.45)

$$y_i(a,\vec{p}) = p_i^{-\kappa}q_i^{\kappa-1}\Omega\sum_{A\,|\,i}\left[\hat{f}(a,A)\left[\mu_i\Gamma(1-\sigma(\kappa-1))\left(\mu_i - \mu_{\text{-}i} + \mu_{\text{-}i}\left(\frac{p_i}{q_i}\right)^{1/\sigma}\sum_{i'\in A}\left(\frac{p_{i'}}{q_{i'}}\right)^{\text{-}1/\sigma}\right)^{\sigma(\kappa-1)-1}\right]\right] \tag{A.47}$$

Rearange and recall the definition of $\bar{\Gamma}$ from (A.29)

$$= \mu_i\bar{\Gamma}^{1-\kappa}p_i^{-\kappa}q_i^{\kappa-1}\Omega\sum_{A\,|\,i}\left[\hat{f}(a,A)\left(\mu_i - \mu_{\text{-}i} + \mu_{\text{-}i}\left(\frac{p_i}{q_i}\right)^{1/\sigma}\sum_{i'\in A}\left(\frac{p_{i'}}{q_{i'}}\right)^{\text{-}1/\sigma}\right)^{\sigma(\kappa-1)-1}\right] \tag{A.48}$$

In the special case where $\mu = \mu_{\text{-}i} = 1$, use (A.29) and (A.48) to get

$$y_i(a, \vec{p}) = \frac{\bar{\Gamma}^{1-\kappa}\Omega}{q_i} \left(\tfrac{p_i}{q_i}\right)^{-1-1/\sigma} \sum_{A\,|\,i} \left[\hat{f}(a, A) \left(\sum_{i' \in A} \left(\tfrac{p_{i'}}{q_{i'}}\right)^{\text{-}1/\sigma} \right)^{\sigma(\kappa-1)-1} \right] \tag{A.49}$$

Reorganize to get (A.30)

$$y_i(a, \vec{p}) = \bar{\Gamma}^{1-\kappa}\Omega\, q_i^{\kappa-1} p_i^{-\kappa} \sum_{A\,|\,i} \left(\hat{f}(a, A) \left[\sum_{i' \in A} \left(\frac{q_i}{q_{i'}} \frac{p_{i'}}{p_i} \right)^{\text{-}1/\sigma} \right]^{\sigma(\kappa-1)-1} \right) \tag{A.50}$$

In the case of identical qualities, normalized to the constant $q = \bar{\Gamma}$, simplify (A.48) to get

$$y_i(a, \vec{p}) = \mu_i p_i^{-\kappa}\Omega \sum_{A\,|\,i} \left[\hat{f}(a, A) \left(\mu_i - \mu_{\text{-}i} + \mu_{\text{-}i} p_i^{1/\sigma} \sum_{i' \in A} p_{i'}^{\text{-}1/\sigma} \right)^{\sigma(\kappa-1)-1} \right] \tag{A.51}$$

$$= \mu_i p_i^{-\kappa}\Omega \sum_{A\,|\,i} \left[\hat{f}(a, A) \left(\mu_i + \mu_{\text{-}i} \sum_{i' \in A \setminus i} \left(\frac{p_{i'}}{p_i} \right)^{\text{-}1/\sigma} \right)^{\sigma(\kappa-1)-1} \right] \tag{A.52}$$

Finally, in the case of $\mu_i = \mu_{\text{-}i} = 1$ and $q_i = \bar{\Gamma}$, simplify (A.48)

$$y_i(a, \vec{p}) = \Omega\, p_i^{-1/\sigma-1} \sum_{A\,|\,i} \left(\hat{f}(a, A) \left[\sum_{i' \in A} p_{i'}^{\text{-}1/\sigma} \right]^{\sigma(\kappa-1)-1} \right) \tag{A.53}$$

□

A.4 Total Demand for Symmetric Firms

Define a measure of sorting

$$\Psi(\mu, \hat{n}) \equiv \mu + (\hat{n} - 1)(2 - \mu) \tag{A.54}$$

With the off-equilibrium version

$$\hat{\Psi}(p_i, p, \mu_i, \mu, \hat{n}) \equiv \Psi(\mu, \hat{n}) + \mu_i \left(\tfrac{p_i}{p}\right)^{\text{-}1/\sigma} - \mu \tag{A.55}$$

Note that this nests (A.54), $\Psi(\mu, \hat{n}) = \hat{\Psi}(p, p, \mu, \mu, \hat{n})$. The following is a generalization of Main Proposition 2 to support the $\mu_i \neq \mu$ case,

Proposition 3 (Total Demand for N Firms with Symmetric Strategies)**.** *Given Assumption 2, the independence of ξ_j and $A_j(a)$, and that every firm $i' \neq i$ chooses the price p, the demand curve from Main Definition 1 faced by firm i choosing p_i is Take symmetric $p, \mu_{\text{-}i}$ for all other firms, then if an equilibrium exists*

$$y(a, p_i, p, \mu_i, \mu_{\text{-}i}) = (1 - f_0(a))\mu_i \frac{p_i^{-\kappa}}{N} \mathbb{E}_a \left[\hat{n} \left(\mu_i + (\hat{n} - 1)\mu_{\text{-}i} \left(\frac{p}{p_i} \right)^{\text{-}1/\sigma} \right)^{\sigma(\kappa-1)-1} \right] \Omega \tag{A.56}$$

Which can be reorganized as

$$y_i(a, p_i, p, \mu_i, \mu) = (1 - f_0(a)) p_i^{-\kappa}\Omega \frac{1}{N} \mu_i^{\sigma(\kappa-1)} \mathbb{E}_a \left[\hat{n} \left(1 + \frac{\mu_{\text{-}i}}{\mu_i} (\hat{n} - 1) \left(\frac{p}{p_i} \right)^{-1/\sigma} \right)^{\sigma(\kappa-1)-1} \right] \tag{A.57}$$

A few special cases (always assuming existence)

1. If $\mu_i = \mu = 1$

$$y(a,p_i,p) = \frac{1-f_0(a)}{N} p_i^{-\kappa} \mathbb{E}_a \left[\hat{n} \left(1 + (\hat{n}-1) \left(\frac{p}{p_i} \right)^{-1/\sigma} \right)^{\sigma(\kappa-1)-1} \right] \Omega \tag{A.58}$$

2. If $\mu_i = \mu = 1$ and $p_i = p$,

$$Ny(a,p) = \underbrace{(1-f_0(a))}_{\substack{\textit{Limited}\\ \textit{Awareness}}} \underbrace{p^{-\kappa}\Omega}_{\substack{\textit{Typical}\\ \textit{CES}}} \underbrace{\mathbb{E}_a \left[\hat{n}^{\sigma(\kappa-1)} \right]}_{\textit{From Sorting}} \tag{A.59}$$

3. If the $\mu = 1$ and the firm is a monoplist,

$$y(a,p) = (1-f_0(a))p^{-\kappa}\Omega \tag{A.60}$$

4. If $p = p_i, \mu \geq 1$ and $\mu_{\text{-}i} = 2 - \mu$ (i.e. symmetric strategies with match distortions)

$$y(a,p,\mu) = (1-f_0(a))\mu_i \frac{p^{-\kappa}}{N} \mathbb{E}_a \left[\hat{n}\Psi(\mu,\hat{n})^{\sigma(\kappa-1)-1} \right] \Omega \tag{A.61}$$

Proof of Proposition 3. This nests the proof of Main Proposition 2. Recall that

$$\mathbb{E}_a [g(\hat{n})] \equiv \sum_{n=1}^{N} \frac{f_n(a)}{1-f_0(a)} g(n) \tag{A.62}$$

Using Main (12) gives sums in terms of moments of the $\hat{n}$ random variable,

$$\sum_{n=1}^{N} f_n(a)g(n) = (1-f_0(a))\mathbb{E}_a [g(\hat{n})] \tag{A.63}$$

First, note that in a symmetric equilibrium, finding the probability that a particular firm is in an awareness set of size n with total N firms is distributed Hypergeometric (i.e., an urn problem without replacement). From this, as there is only one possible successful state, the pmf of the Hypergeometric evaluated at the successful state is $\frac{\binom{N-1}{n-1}}{\binom{N}{n}} = \frac{n}{N}$.

Given N firms and symmetric evolution of awareness, the mass of consumers aware of firm i who have a total awareness set of size n is, then, $\frac{n}{N} f_n(a)$. Simplify (A.30) with $q_i = \bar{\Gamma}$, the price of the firm p_i, and the symmetric price of all other firms p,

$$y(a,p_i,p) = \frac{p_i^{-\kappa}}{N} \left[\sum_{n=1}^{N} f_n(a) n \left(1 + (n-1) \left(\frac{p}{p_i} \right)^{\text{-}1/\sigma} \right)^{\sigma(\kappa-1)-1} \right] \Omega \tag{A.64}$$

Simplify with (A.63) to get Main (13). Substitution of $p_i = p$ gives Main (14)

Symmetric match distortion policies Consider an equilibrium where all $\mu_i = \mu \geq 1$. If all draws are distorted in this way symmetrically above 1 for providing good matches, then a simple countervailing distortion for the other firms in the awareness sets is the symmetric

$$\mu_{\text{-}i} = 1 - (\mu - 1) = 2 - \mu \tag{A.65}$$

With this, define a funtion determining the distortion of the effective awareness set sizes as

$$\Psi(\mu,\hat{n}) \equiv \mu + (\hat{n}-1)\mu_{\text{-}i} \tag{A.66}$$

$$= \mu + (\hat{n}-1)(2-\mu) \tag{A.67}$$

Note that in the undistorted case of $\mu = 1$, $\Psi(1,\hat{n}) = \hat{n}$

Demand with symmetric policies and asymmetric match distributions In this case, assume all firms have identical μ_i and that maintain the assumption that the prices for other firms are p and the price for the individual firm i is p, then from (A.31)

$$y(a, p_i, p, \mu_i, \mu_{\text{-}i}) = \mu_i p_i^{-\kappa}\Omega \sum_{A\,|\,i}\left[\hat{f}(a,A)\left(\mu_i + \mu_{\text{-}i}\sum_{i'\in A\setminus i}\left(\frac{p}{p_i}\right)^{\text{-}1/\sigma}\right)^{\sigma(\kappa-1)-1}\right] \tag{A.68}$$

$$= \frac{\mu_i}{N}\Omega p_i^{-\kappa}\sum_{n=1}^{N}\left[f_n(a)n\left(\mu_i + (n-1)\mu_{\text{-}i}\left(\frac{p}{p_i}\right)^{\text{-}1/\sigma}\right)^{\sigma(\kappa-1)-1}\right] \tag{A.69}$$

$$= \mu_i\frac{p_i^{-1-1/\sigma}}{N}\left[\sum_{n=1}^{N} f_n(a)n\left(\mu_i p_i^{\text{-}1/\sigma} + (n-1)\mu_{\text{-}i}p^{\text{-}1/\sigma}\right)^{\sigma(\kappa-1)-1}\right]\Omega \tag{A.70}$$

$$= (1-f_0(a))\mu_i\frac{p_i^{-1-1/\sigma}}{N}\mathbb{E}_a\left[\hat{n}\left(\mu_i p_i^{\text{-}1/\sigma} + (\hat{n}-1)\mu_{\text{-}i}p^{\text{-}1/\sigma}\right)^{\sigma(\kappa-1)-1}\right] \tag{A.71}$$

Note that in the case of $p_i = p$ and with symmetric μ,

$$y(a,p,\mu) = (1-f_0(a))\mu_i\frac{p^{-\kappa}}{N}\mathbb{E}_a\left[\hat{n}\left(\mu_i + (\hat{n}-1)\mu_{\text{-}i}\right)^{\sigma(\kappa-1)-1}\right]\Omega \tag{A.72}$$

$$= (1-f_0(a))\mu_i\frac{p^{-\kappa}}{N}\mathbb{E}_a\left[\hat{n}\Psi(\mu,\hat{n})^{\sigma(\kappa-1)-1}\right]\Omega \tag{A.73}$$

□

A.5 Prices for Symmetric Firms

Define age-dependent markup under asymetric match distributions as a generalization of Main (16) as,

$$\Upsilon(a|\mu) \equiv 1 + \sigma\left[1 - \mu(1-\sigma(\kappa-1))\frac{\mathbb{E}_a\left[\hat{n}\Psi(\mu,\hat{n})^{\sigma(\kappa-1)-2}\right]}{\mathbb{E}_a\left[\hat{n}\Psi(\mu,\hat{n})^{\sigma(\kappa-1)-1}\right]}\right]^{-1} \tag{A.74}$$

which nests the version with $\mu = 1$,

$$\Upsilon(a) \equiv \Upsilon(a|1) =\equiv 1 + \sigma\left[1 - (1-\sigma(\kappa-1))\frac{\mathbb{E}_a\left[\hat{n}^{\sigma(\kappa-1)-1}\right]}{\mathbb{E}_a\left[\hat{n}^{\sigma(\kappa-1)}\right]}\right]^{-1} \in [1+\sigma, \frac{\kappa}{\kappa-1}] \tag{A.75}$$

The following is a generalization of Main Proposition 3,

Proposition 4 (Pricing Strategy Taking Symmetric Strategies as Given)**.** *In cases where an equilibrium exists and all firms have symmetric strategies, i.e.* $p_i = p$ *and* $\mu_i = \mu$ *with marginal cost* mc

$$p(a|\mu) = \Upsilon(a|\mu)mc \tag{A.76}$$

The off-equilibrium pricing strategy for $p_i(\mu_i, a, p, \mu)$ *is the solution to*

$$0 = (-p_i + mc + \sigma\, mc)g(p_i,p,\mu_i,\mu) + p_i(p_i - mc)\sigma\boldsymbol{\partial}_{p_i}g(p_i,p,\mu_i,\mu) \tag{A.77}$$

Where

$$g(p_i,p,\mu_i,\mu) \equiv p^{1/\sigma-\kappa+1}\mathbb{E}_a\left[\hat{n}\hat{\Psi}(p_i,p,\mu_i,\mu,\hat{n})^{\sigma(\kappa-1)-1}\right] \tag{A.78}$$

$$\boldsymbol{\partial}_{p_i}g(p_i,p,\mu_i,\mu) \equiv \frac{\mu_i(1-\sigma(\kappa-1))}{\sigma}p^{1/\sigma-\kappa}\left(\frac{p_i}{p}\right)^{-1-1/\sigma}\mathbb{E}_a\left[\hat{n}\hat{\Psi}(p_i,p,\mu_i,\mu,\hat{n})^{\sigma(\kappa-1)-2}\right] \tag{A.79}$$

Proof of Proposition 4. Assume a symmetric price p, and use a variation of Main (13) with (A.31) to form the optimization problem

$$p_i = \arg\max_{\tilde{p}\geq 0} \left\{ (\tilde{p} - mc)\mu_i \frac{1 - f_0(a)}{N} \tilde{p}^{-1-1/\sigma} \mathbb{E}_a \left[\hat{n} \left(\mu_i \tilde{p}^{-1/\sigma} + (\hat{n} - 1)\mu_{\text{-}i} p^{-1/\sigma} \right)^{\sigma(\kappa-1)-1} \right] \Omega \right\} \tag{A.80}$$

Define

$$g(p_i, p) \equiv \mathbb{E}_a \left[\hat{n} \left(\mu_i p_i^{-1/\sigma} + (\hat{n} - 1)\mu_{\text{-}i} p^{-1/\sigma} \right)^{\sigma(\kappa-1)-1} \right] \tag{A.81}$$

Assume existence and take the first-order condition of (A.80) with respect to p_i

$$0 = (-p_i + mc + \sigma\, mc) g(p_i, p) + p_i(p_i - mc)\sigma \frac{\partial g(p_i, p)}{\partial p_i} \tag{A.82}$$

Note that for the symmetric price $p_i = p$,

$$g(p, p) = p^{1/\sigma-\kappa+1} \mathbb{E}_a \left[\hat{n}(\mu_i + \mu_{\text{-}i}(\hat{n} - 1))^{\sigma(k-1)-1} \right] \tag{A.83}$$

Take the partial of $g(p_i, p)$ and evaluate at $p = p_i$

$$\boldsymbol{\partial}_{p_i} g(p, p) = \frac{\mu_i(1 - \sigma(\kappa - 1))}{\sigma} p^{1/\sigma-\kappa} \mathbb{E}_a \left[\hat{n}(\mu_i + \mu_{\text{-}i}(\hat{n} - 1))^{\sigma(\kappa-1)-2} \right] \tag{A.84}$$

Evaluate at p and substitute into (A.82)

$$0 = -p + mc + \sigma\, mc + \mu_i(p - mc)(1 - \sigma(\kappa - 1)) \frac{\mathbb{E}_a \left[\hat{n}(\mu_i + \mu_{\text{-}i}(\hat{n} - 1))^{\sigma(k-1)-2} \right]}{\mathbb{E}_a \left[\hat{n}(\mu_i + \mu_{\text{-}i}(\hat{n} - 1))^{\sigma(k-1)-1} \right]} \tag{A.85}$$

Finally, solve for the price, and substitute with $\Psi(\mu, \hat{n})$ from (A.54) for the symmetric solution

$$p(a|\mu) = \left(1 + \sigma \left[1 - \mu(1 - \sigma(\kappa - 1)) \frac{\mathbb{E}_a \left[\hat{n}\Psi(\mu, \hat{n})^{\sigma(\kappa-1)-2} \right]}{\mathbb{E}_a \left[\hat{n}\Psi(\mu, \hat{n})^{\sigma(\kappa-1)-1} \right]} \right]^{-1} \right) mc \tag{A.86}$$

$$= \Upsilon(a|\mu) mc \tag{A.87}$$

Consider the off-equilibrium pricing strategy given a μ, p from every other firm, but a single firm (not large enough to change aggregates) deviates with a $\mu_i \neq \mu$ and $p_i \neq p$. First, define take derivatives of (A.55)

$$\boldsymbol{\partial}_{p_i} \hat{\Psi}(p_i, p, \mu_i, \mu, \hat{n}) = \frac{-1}{\sigma} \frac{\mu_i}{p} \left(\frac{p_i}{p} \right)^{-1-1/\sigma} \tag{A.88}$$

$$\boldsymbol{\partial}_{\mu_i} \hat{\Psi}(p_i, p, \mu_i, \mu, \hat{n}) = \left(\frac{p_i}{p} \right)^{-1/\sigma} \tag{A.89}$$

Rearrange (A.81) to get

$$g(p_i, p, \mu_i, \mu) \equiv \mathbb{E}_a \left[\hat{n} \left(\mu_i p_i^{-1/\sigma} + (\hat{n} - 1)\mu_{\text{-}i} p^{-1/\sigma} \right)^{\sigma(\kappa-1)-1} \right] \tag{A.90}$$

Note in the fully symmetric $p_i = p, \mu_i = \mu$ that $\hat{\Psi}(p_i, p, \mu_i, \mu, \hat{n}) = \Psi(\mu, \hat{n})$. With (A.55) and (A.90)

$$g(p_i, p, \mu_i, \mu) = p^{1/\sigma-\kappa+1} \mathbb{E}_a \left[\hat{n}\hat{\Psi}(p_i, p, \mu_i, \mu, \hat{n})^{\sigma(k-1)-1} \right] \tag{A.91}$$

Take the partial with respect to p_i

$$\boldsymbol{\partial}_{p_i} g(p_i, p, \mu_i, \mu) = (\sigma(\kappa - 1) - 1)p^{1/\sigma - \kappa + 1}\mathbb{E}_a\left[\hat{n}\hat{\Psi}(p_i, p, \mu_i, \mu, \hat{n})^{\sigma(k-1)-2}\boldsymbol{\partial}_{p_i}\hat{\Psi}(p_i, p, \mu_i, \mu, \hat{n})\right] \tag{A.92}$$

Use (A.88)

$$\boldsymbol{\partial}_{p_i} g(p_i, p, \mu_i, \mu) = \frac{\mu_i(1 - \sigma(\kappa - 1))}{\sigma}p^{1/\sigma - \kappa}\left(\frac{p_i}{p}\right)^{-1-1/\sigma}\mathbb{E}_a\left[\hat{n}\hat{\Psi}(p_i, p, \mu_i, \mu, \hat{n})^{\sigma(\kappa-1)-2}\right] \tag{A.93}$$

Note that if $p_i = p$, that this nests (A.84). Go back to (A.82) and substitute with (A.78) and (A.79) without assuming symmetry of the strategy,

$$0 = (-p_i + mc + \sigma\, mc)g(p_i, p, \mu_i, \mu) + p_i(p_i - mc)\sigma\boldsymbol{\partial}_{p_i} g(p_i, p, \mu_i, \mu) \tag{A.94}$$

This equation does not have a closed-form solution for p_i, except in the case of $p_i = p$, so implicit solutions are needed to determine the off-equilibrium price strategy.

$\square$

Limits for symmetric pricing and match distributions Given the symmetric pricing equilibria, find asymptotic prices by taking limits as $a \to 0, a \to \infty$, etc.

Proposition 5 (Asymptotic Properties of Prices)**.** *Define $p(a, N)$ as the equilibrium prices conditional on an industry with N firms. For symmetric firms, if a pure-strategy equilibrium exists and the stochastic process has $\lim_{a\to\infty}\mathbb{E}_a[\hat{n}] = N$, then: (1) $p(\infty, N) \equiv \lim_{a\to\infty} p(a) = \frac{(N-1+\sigma(N\kappa-1))}{N-1+\sigma(\kappa-1)}mc < p_1(N)$; (2) $p(a, 1) = p(0, N) = \frac{mc}{\varsigma}$; and (3) $p(\infty, \infty) \equiv \lim_{N\to\infty} p_\infty = (1 + \sigma)mc$*

Proof. The $\lim_{a\to 0}$ price uses the first-order expansion of the counting process for any $\mathbb{Q}$. Regardless of the particular Markov chain, after an infinitesimal amount of time, the support of n will be 0 or 1.

$$p = \arg\max_{\tilde{p}\geq 0}\left\{(\tilde{p} - mc)\tilde{p}^{-1-1/\sigma}\left(\tilde{p}^{-1/\sigma}\right)^{\sigma(\kappa-1)-1}\right\} = \arg\max_{\tilde{p}\geq 0}\left\{(\tilde{p} - mc)p^{-\kappa}\right\} = \frac{mc}{\varsigma} \tag{A.95}$$

For any $\mathbb{Q}$ with a stationary distribution of $f(\infty) = \begin{bmatrix}0 & \dots & 0 & 1\end{bmatrix}$, use $\lim_{a\to\infty}\frac{\mathbb{E}_a\left[\hat{n}^{\sigma(k-1)-1}\right]}{\mathbb{E}_a\left[\hat{n}^{\sigma(k-1)}\right]} = \frac{1}{N}$ and substitute into Main (19). $\square$

A.6 Planning Problem

Consider a problem where the planner can choose the allocation of real output allocated to the different products for the consumer–constrained by the consumer's choice sets and match values. Since we will be considering equilibria where all consumer's have equivalent price indices,

Proposition 6 (Planning Problem with Differentiated Firms)**.** *Take an allocation of Y_j units of aggregate output and Q_j of a quality index to consumer j,*

1. *The planner chooses product i and no others if and only if*

$$\log\left(\frac{1}{q_{i'}}\right) - \log\left(\frac{1}{q_i}\right) > \sigma\left(\xi_{i'j} - \xi_{ij}\right), \quad \forall\, i' \in A_j \setminus \{i\} \tag{A.96}$$

2. *The intensive demand for product i is*

$$y_{ij}(a, \xi_{ij}) = q_i^{\kappa-1}e^{\sigma(\kappa-1)\xi_{ij}}Q_j^\kappa Y_j, \quad y_{i'j} = 0, \forall\, i' \in A_j \setminus \{i\} \tag{A.97}$$

3. With the quality index defined as

$$Q_j \equiv \left(\int_{\mathcal{M}_j} \left(e^{\sigma \xi_m} q_m \right)^{\kappa - 1} \mathrm{d}m \right)^{\frac{1}{1-\kappa}} \tag{A.98}$$

Furthermore, the allocations of the competitive equilibrium and the the planning problem are identical whenever p_{im} is constant for all products and industries (i.e. any constant markup pricing).

Proof of Proposition 6. The planner gives an allocation of $Y_j Q_j$ units of aggregate quality adjusted outputto consumer j, and uses it to choose the appropriate mix of y_{imj} for each consumer - conditional on the equivalent condition here is that they are all given the same proportion of aggregate production for their consumption.

$$\begin{aligned} \max_{y_{imj} \geq 0} & \left(\int_0^M \left(\sum_{i \in A_{mj}} q_{im} e^{\sigma \xi_{imj}} y_{imj} \right)^{\varsigma} \mathrm{d}m \right)^{1/\varsigma} \\ \text{s.t.} & \int_0^M \left[\sum_{i \in A_{mj}} y_{imj} \right] \mathrm{d}m \leq Y_j Q_j \end{aligned} \tag{A.99}$$

The simple linear constraint comes from the fact that all firms have the same normalized CRS productivity, so the planner could just as easily allocate labor to each firm instead to support the consumer's production.

First, define the Lagrangian for the planner's optimization problem in (A.99) with $\lambda_j > 0$ as the Lagrange multiplier on the budget constraint, and $\mu_{imj} \geq 0$ as the Lagrange multipliers ensuring weak positivity on every choice of y_{imj}.

$$\mathcal{L} = \left(\int_0^M \left(\sum_{i' \in A_m} q_{i'm} e^{\sigma \xi_{i'mj}} y_{i'mj} \right)^{\varsigma} \mathrm{d}m \right)^{1/\varsigma} - \lambda_j \left(\int_0^M \sum_{i' \in A_m} y_{i'mj} \mathrm{d}m - Y_j Q_j \right) + \int_0^M \sum_{i' \in A_m} \mu_{i'mj} y_{i'mj} \mathrm{d}m \tag{A.100}$$

The first-order necessary conditions with respect to y_{imj} are

$$S_j \left(\sum_{i' \in A_m} e^{\sigma \xi_{i'mj}} q_{i'm} y_{i'mj} \right)^{\varsigma - 1} q_{im} e^{\sigma \xi_{imj}} = \lambda_j - \mu_{imj} \tag{A.101}$$

$$\lambda_j > 0, \quad \mu_{imj} \geq 0, \quad \mu_{imj} y_{imj} = 0, \quad \forall\, i, m \tag{A.102}$$

where S_j is the same as that of the consumer's problem, i.e. (A.9)
Intensive Demand: Maintain Assumption 2 throughout. Assume, to be verified, that if $|A_m| > 0$, the consumer will almost certainly consume a single product per industry.

To solve for λ_j and the price index, I will follow standard CES algebra under the assumption of consuming, at most, one product per industry. The i index is dropped since there is only one good per industry, and the j index is dropped for simplicity—though given the assumptions for the aggregate consumer, I show in Appendix B.3 that S_j is identical for all j. From (A.101), for industries with non-empty awareness sets,

$$S_j \left(e^{\sigma \xi_m} q_m \right)^{\varsigma} y_m^{\varsigma - 1} = \lambda_j \tag{A.103}$$

Take the ratio of two industries m' and m with positive demand

$$\frac{y_{m'}^{\varsigma-1}}{y_m^{\varsigma-1}} = \left(\frac{e^{\sigma\xi_{m'}}q_{m'}}{e^{\sigma\xi_m}q_m}\right)^{-\varsigma} \tag{A.104}$$

Rearrange,

$$y_{m'} = y_m\left(e^{\sigma\xi_m}q_m\right)^{1-\kappa}\left(e^{\sigma\xi_{m'}}q_{m'}\right)^{\kappa-1} \tag{A.105}$$

Integrate over all industries with positive consumption, $m' \in \mathcal{M}_j$,

$$\int_{\mathcal{M}_j} y_{m'}\mathrm{d}m' = y_m\left(e^{\sigma\xi_m}q_m\right)^{1-\kappa}\int_{\mathcal{M}_j}\left(e^{\sigma\xi_{m'}}q_{m'}\right)^{\kappa-1}\mathrm{d}m' \tag{A.106}$$

Recognize that industry m is infinitesimal, so the integrals are identical with or without industry m. Hence, a quality index cannot effect the price index either through a change in the intensive demand or by switching between different i in $\mathcal{I}_m$. Define the quality index as

$$Q_j \equiv \left(\int_{\mathcal{M}_j}\left(e^{\sigma\xi_m}q_m\right)^{\kappa-1}\mathrm{d}m\right)^{\frac{1}{1-\kappa}} \tag{A.107}$$

Reorganize (A.106) using the price index, noting that the left-hand side of the equality is the budget

$$y_m = \left(e^{\sigma\xi_m}q_m\right)^{\kappa-1}{Q_j}^{\kappa}Y_j \tag{A.108}$$

Extensive Demand: The proof strategy is to assume that a single product is consumed, to use the non-negativity of the Lagrange multipliers for the other products to determine a set of inequalities necessary for this choice to hold, and then to show that multiple products will be chosen only under measure 0 events. The inequality constraints in (A.7) for all products $i, i' \in \mathcal{I}_m$ give

$$S_j\left(e^{\sigma\xi_i}q_iy_i\right)^{\varsigma-1}q_{i'}e^{\sigma\xi_{i'}} \leq \lambda_j \tag{A.109}$$

Rearrange (A.103)

$$\left(y_iq_ie^{\sigma\xi_i}\right)^{\varsigma-1} = \frac{\lambda_j}{S_j}\frac{1}{q_ie^{\sigma\xi_i}} \tag{A.110}$$

Combine with (A.109)

$$\tfrac{\lambda_j}{S_j}\tfrac{1}{q_i}e^{-\sigma\xi_{ij}}q_{i'}e^{\sigma\xi_{i'j}} \leq \tfrac{\lambda_j}{S_j} \tag{A.111}$$

Take logs and rearrange

$$\log\left(\tfrac{1}{q_{i'}}\right) - \log(\tfrac{1}{q_i}) \geq \sigma(\xi_{i'j} - \xi_{ij}) \tag{A.112}$$

To finish off the proof, showing that when prices are a constant markup the competitittive equilibrium generates the planners allocation. For any constant K (interpreted as a constant markup) let $\hat{p}_im = K$, then use (A.5) and (A.107) to show that $P_j = KQ_j$.

Finally, note that if the real price is $p_i = \hat{p_{im}}/P$, then in that case, $p_i = Q_j^{-1}$. Substitute into (A.15) to complete the comparison, if $Y_j = \Omega_j$

Finally, note that the extensive demand in (A.3) lines up as well when $\hat{p}$ are all constant since they drop out of the inequality, and it becomes (A.112) $\square$

B Fully Dynamic Model and Aggregation

This section provides proofs for the aggregation to a neoclassical growth economy with an awareness-dependent wedge. It relies on the derivations for markups, output, etc. in Appendix A.

B.1 Firm Value and Industry Equilibrium

Define the prices of all of the other firms in the industry at age a as $p_{\text{-}i}(a) \equiv \{p_{i'}(a) | i' \in \mathcal{I} \neq i\}$. Using the total demand derived in Main (13), firm i's value at age a is the present discounted value (PDV) of profits. The following is the dynamic value of the firm given the sequence of prices:

Definition 2 (Firm Value)**.** *Given a discount rate $r > 0$ and prices $p(a)$, the firm's valuation is the present discounted value of profits*

$$v(a, p_i, p_{\text{-}i}) \equiv \int_0^\infty e^{-r(a+\tau)} \left[\underbrace{(p_i(a+\tau) - mc(a+\tau)) y_i(p_i(a+\tau), p_{\text{-}i}(a+\tau))}_{\equiv \pi_i(a+\tau),\ \textit{Profits}} \right] \mathrm{d}\tau \tag{B.1}$$

Given the value, I can define a standard pure-strategy symmetric equilibrium for a given industry:

Definition 3 (Symmetric Industry Equilibrium)**.**
A (post-entry) symmetric industry equilibrium with history-independent prices is a set of: (1) demand functions $y(a, p, A, \xi) \to \mathbb{R}$; (2) firm pricing functions $p(a)$; and (3) evolving distributions of the consumer awareness count $f(a)$, such that: (a) given $p(a)$, $y(\cdot)$ is optimal for the consumer according to Main (6)*; (b) given $y(\cdot)$ and the aggregation in Main* (13)*, $p(a)$ are a symmetric pure strategy BNE of the game as in Main Definition 2; and (c) $f(a)$ evolves according to the law of motion for the awareness process.*

B.2 Static Industry Conditions

Proof. The following starts with a standard derivation of the marginal cost of a Cobb-Douglas production function, and then applies the heterogeneous markups and sorting quality. The cost minimization problem is

$$\min_{\ell, K} \{rK + w\ell\} \text{ s.t. } y = zK^\alpha \ell^{1-\alpha} \tag{B.2}$$

The first-order conditions for the cost minimization in (B.2) are

$$r = \lambda z \alpha \frac{y}{K} \tag{B.3}$$

$$w = \lambda z (1-\alpha) \frac{y}{L} \tag{B.4}$$

Combine the FONCs and define k to find the optimal capital-labor ratio for all firms

$$k \equiv \frac{K}{\ell} = \frac{\alpha}{1-\alpha} \frac{w}{r} \tag{B.5}$$

From the production technology,

$$\ell = k^{-\alpha} \frac{y}{z} \tag{B.6}$$

Substitute (B.5) and (B.6) into the total cost and simplify

$$rK + w\ell = \frac{1}{1-\alpha} w\ell = \frac{k^{-\alpha} wy}{(1-\alpha)z} \tag{B.7}$$

Taking $\boldsymbol{\partial}_y$ gives the constant marginal cost,

$$mc = \tfrac{1}{1-\alpha} z^{-1} k^{-\alpha} w \tag{B.8}$$

Substitute Main (15), (16) and (19), into Main (17) and (18), and factor into age and time dependent components,

$$Y(t,a) = (1 - f_0(a))\Upsilon(a)^{-\kappa} q(a)\, mc(t)^{-\kappa} \Omega(t) \tag{B.9}$$

$$\Pi(t,a) = (1 - f_0(a))(\Upsilon(a) - 1)\Upsilon(a)^{-\kappa} q(a)\, mc(t)^{1-\kappa} \Omega(t) \tag{B.10}$$

Use (B.6), (B.8) and (B.9) to find labor demand by industry age

$$L(t,a) = (1 - f_0(a))\Upsilon(a)^{-\kappa} q(a)\ (1-\alpha) w(t)^{-1} mc(t)^{1-\kappa} \Omega(t) \tag{B.11}$$

The capital used by the industry of age a comes from (B.5) and (B.11), which is also the book value of tangible assets

$$K(t,a) = k(t) L(t,a) \tag{B.12}$$

At this point, I have collapsed all industry-specific functions into the proportion of consumers unaware of any firm $1 - f_0(a)$, the markup $\Upsilon(a)$ and quality $q(a)$. The aggregate contributions to prices, profits, and output are the components of marginal cost (i.e., $k(t)$, real wages $w(t)$, aggregate productivity $z(t)$) and real income $\Omega(t)$). $\square$

B.3 Price Index and Aggregation

Proof of Main Proposition 4. First, derive the price index, Main (7), in terms of the age distribution of industries. Given Assumption 1 and that all consumers were alive at the birth of every industry, the price index will be identical for all consumers.[2] Take Main (7), temporarily drop the t index where appropriate for clarity, and denote the age of industry m as $a(m)$,

$$P_j(\xi) = \bar{\Gamma}^{-1} \left(\int_{\mathcal{M}_j} e^{\sigma(\kappa-1)\xi_{mj}} \hat{p}(a(m))^{1-\kappa} \mathrm{d}m \right)^{\frac{1}{1-\kappa}} \tag{B.13}$$

While the price in equilibrium is only a function of age, I need to take into account the idiosyncratic $\mathcal{M}_j$ and matches based on ξ. As derived from $\mathbb{Q}$, the proportion of firms of age a that a consumer is aware of (i.e., $n > 0$) is $1 - f_0(a)$. Hence, given the unnormalized cdf of industry age, $M(t)\Phi(t,a)$, I can replace $\mathcal{M}_j$ with an integral over the age distribution weighted by the proportion that they have an awareness of.

While the price is directly a function of age, the idiosyncratic ξ_{mj} match value can be shown to be a function of age in expectation. Independent of age, recall that in the symmetric equilibrium, the consumer chooses the product with the highest match value given an awareness set of size n. If the ξ_{imj} are independent, then the distribution of the maximum of n draws from the $g(\xi)$ distribution

[2] These assumptions can be relaxed if additional consumer state variables in the $\hat{\Psi}(\cdot)$ distribution. For example, if agents are born and die at different times, the price index would be consumer age-dependent, as well. However, consumers born in the same year would have identical pricing indices. Finally, as consumers would be entering and exiting, the evolution of $f(a)$ in Main (1) would need to be modified since it is the distribution conditional on survival.

is the first order-statistic, $g_{(n)}(\xi)$. Industry age enters the matches through a distribution of n for the continuum of industries of a particular age. Convert to the $\hat{n}$ random variable and use (A.63),

$$P = \bar{\Gamma}^{-1}\left(\int (1 - f_0(a))\hat{p}(a)^{1-\kappa}\mathbb{E}_a\left[\int_{-\infty}^{\infty} e^{\sigma(\kappa-1)\xi} g_{(\hat{n})}(\xi)\mathrm{d}\xi\right] M\mathrm{d}\Phi(t,a)\right)^{\frac{1}{1-\kappa}} \tag{B.14}$$

For the Gumbel distribution, the order-statistic for the maximum of $\hat{n}$ draws is also Gumbel due to max-stability,

$$g_{(\hat{n})}(\xi) = \hat{n}e^{-\xi}e^{-\hat{n}e^{-\xi}} \tag{B.15}$$

From this, the following integral is calculated using $\int_{-\infty}^{\infty} e^{-Ax}e^{-Be^{-x}}\mathrm{d}x = B^{-A}\Gamma(A)$:

$$\int_{-\infty}^{\infty} e^{\sigma(\kappa-1)\xi} g_{(\hat{n})}(\xi)\mathrm{d}\xi = \Gamma(1 - \sigma(\kappa-1))\hat{n}^{\sigma(\kappa-1)} \tag{B.16}$$

Substitute into (B.14) and use definitions for $\bar{\Gamma}$ and Main (15) to find the price index in Main (21)

$$P(t) = \left(\int (1 - f_0(a))\hat{p}(t,a)^{1-\kappa}q(a)M\mathrm{d}\Phi(t,a)\right)^{\frac{1}{1-\kappa}} \tag{B.17}$$

Divide both sides by $P(t)$, substitute from Main (19) and (22), use $\hat{p} \equiv pP$, and reorganize for $mc(t)$,

$$mc(t) = M(t)^{\frac{1}{\kappa-1}}Q(t) \tag{B.18}$$

Use Main (25) to solve for $w(t)$ and then use Main (24)

$$w(t) = (1-\alpha)k(t)^{\alpha}z(t)M(t)^{\frac{1}{\kappa-1}}Q(t) = (1-\alpha)Z(t)k(t)^{\alpha}B(t) \tag{B.19}$$

Composite Good: From Main (3), define the composite good,

$$Y_j \equiv \left(\int_0^M \left(\sum_{i\in A_{mj}} \bar{\Gamma}e^{\sigma\xi_{imj}}y_{imj}\right)^{\varsigma} \mathrm{d}m\right)^{1/\varsigma} \tag{B.20}$$

Use the same approach to grouping as in (B.13)

$$Y_j(\xi)^{\varsigma} = \bar{\Gamma}^{\varsigma}\int_{\mathcal{M}_j} \left(e^{\sigma\xi_m}y(a,\xi_m)\right)^{\varsigma}\mathrm{d}m \tag{B.21}$$

From Main (6)

$$= \bar{\Gamma}^{\varsigma}\int_{\mathcal{M}_j} \left(e^{\sigma\xi_m}\bar{\Gamma}^{\kappa-1}e^{\sigma(\kappa-1)\xi_m}p(a)^{-\kappa}\Omega\right)^{\varsigma}\mathrm{d}m = \Omega^{\varsigma}\bar{\Gamma}^{\kappa-1}\int_{\mathcal{M}_j} e^{\sigma(\kappa-1)\xi_m}p(a)^{1-\kappa}\mathrm{d}m \tag{B.22}$$

Use $\hat{p} \equiv p/P$ and reorganize

$$= \Omega^{\varsigma}P^{\kappa-1}\frac{\int_{\mathcal{M}_j} e^{\sigma(\kappa-1)\xi_m}\hat{p}(a)^{1-\kappa}\mathrm{d}m}{\bar{\Gamma}^{1-\kappa}} \tag{B.23}$$

Combine (B.13) and (B.23) and simplify to find that $Y(t) = \Omega(t)$. Hence, the composite good $Y(t)$ acts as an aggregate good and is equal to real income. This is a standard result from CES preferences and monopolistic competition, and it generalizes here. With this, I can write the consumer's dynamic and labor supply problems as those of a representative agent and representative firm (conditional on an agent distribution), with TFP given by Main (24).

Income and Aggregation Production Take (B.11) and aggregate to find labor demand

$$L(t) = \int_0^\infty L(t,a)M(t)\mathrm{d}\Phi(t,a) \tag{B.24}$$

$$= (1-\alpha)M(t)w(t)^{-1}mc(t)^{1-\kappa}Y(t)\int_0^\infty (1-f_0(a))\Upsilon(a)^{-\kappa}q(a)\mathrm{d}\Phi(t,a) \tag{B.25}$$

Use (B.18) and reorganize

$$w(t)L(t) = (1-\alpha)Y(t)Q(t)^{1-\kappa}\int_0^\infty (1-f_0(a))\Upsilon(a)^{-\kappa}q(a)\mathrm{d}\Phi(t,a) \tag{B.26}$$

With Main (22) and (23),

$$w(t)L(t) = (1-\alpha)B(t)Y(t) \tag{B.27}$$

Substitute from (B.19) and Main (24) into (B.27), and reorganize to get physical output as a function of aggregates and labor supply

$$Y(t) = Z(t)L(t)k(t)^{\alpha} \tag{B.28}$$

From (B.27), note that the labor share of output, $\frac{P(t)w(t)L(t)}{P(t)Y(t)}$, is $(1-\alpha)B(t)$, with $\alpha B(t)$ going to capital, and $1-B(t)$ going to profits. In the case of monopolistic competition, for any age and quality distribution, $B(t) = (\kappa-1)/\kappa$ so that the profit share is constant at $1/\kappa$.

Profits and Aggregate Value Substitute (B.18) and Main (22)into (B.10)

$$\Pi(t,a) = (1-f_0(a))(\Upsilon(a)-1)\Upsilon(a)^{-\kappa}q(a)M(t)^{-1}Q(t)^{1-\kappa}Y(t) \tag{B.29}$$

Aggregate profits and use Main (22)

$$\Pi(t) = \int \Pi(t,a)M(t)\mathrm{d}\Phi(t,a) = (1-B(t))Y(t) \tag{B.30}$$

Aggregate PDV of profits in a stationary economy (without installed capital),

$$V = \frac{1-B}{1-r}Y \tag{B.31}$$

The aggregate Tobin's Q is the market to book value (i.e., (PDV profits + book value)/book value, where the book value is k due to the price normalization

$$\text{Tobin's Q} = 1 + \frac{1-B}{1-r}\frac{Y}{k} \tag{B.32}$$

Industry Profits and Allocations Calculate the valuation of an entire industry *at entry* in a stationary economy, from (B.1), (B.18) and (B.28) to (B.29),

$$V(t,a) = ZQ^{1-\kappa}M^{-1}k^{\alpha}\int_0^\infty e^{-r\tau}\left[(1-f_0(\tau+a))q(\tau+a)(\Upsilon(\tau+a)-1)\Upsilon(\tau+a)^{-\kappa}\right]\mathrm{d}\tau \tag{B.33}$$

To find the book value, take (B.11) to (B.12), (B.18), (B.19) and (B.28),

$$K(t,a) = (1-f_0(a))\Upsilon(a)^{-\kappa}q(a)\,B^{-1}M^{-1}Q^{1-\kappa}k \tag{B.34}$$

From (B.33) and (B.34), Tobin's Q (i.e., (PDV of Profits + replacement cost of capital)/(replacement cost of capital) of an industry of age a is

$$\text{Tobin's Q}(a) \equiv 1 + \frac{BY}{k}\frac{\int_0^\infty e^{-r\tau}\left[(1-f_0(\tau+a))q(\tau+a)(\Upsilon(\tau+a)-1)\Upsilon(\tau+a)^{-\kappa}\right]\mathrm{d}\tau}{(1-f_0(a))\Upsilon(a)^{-\kappa}q(a)} \tag{B.35}$$

$\square$

B.4 Aggregation with Asymmetric Match Distributions

As a variation on Appendix B.3, consider cases where $\mu > 1$ with a symmetric strategy (i.e. $\mu_i = \mu$ and $p_i = p$)

While the price is directly a function of age, the idiosyncratic ξ_{mj} match value can be shown to be a function of age in expectation. Independent of age, recall that in the symmetric price equilibrium, the consumer chooses the product with the highest match value given an awareness set of size n.

The key calculation, then, is to determine what the distribution of the maximum ξ draw as a function of the awareness set sizes, which we denote with CDF $G_{(\hat{n})}(\xi)$ and pdf $g_{(\hat{n})}(\xi)$. As a simplest case, when $\mu = \mu_{\text{-}i} = 1$ the distribution is simply the first order-statistic for the maximum of $\hat{n}$ draws–also Gumbel due to max-stability,

$$g_{(\hat{n})}(\xi) = \hat{n}e^{-\xi}e^{-\hat{n}e^{-\xi}} \tag{B.36}$$

In the more complicated case of a single draw centered at μ and other draws centered at $\mu_{\text{-}i}$, the product of the CDFs is modified,

$$\begin{aligned} G_{(\hat{n})}(\xi) &= e^{-\mu e^{-\xi}}\left(e^{-\mu_{\text{-}i}e^{-\xi}}\right)^{\hat{n}-1} &\text{(B.37)}\\ &= e^{-(\mu_i+(\hat{n}-1)\mu_{\text{-}i})e^{-\xi}} &\text{(B.38)}\end{aligned}$$

Differentiate to find the pdf, which nests the simpler case when $\mu = \mu_{\text{-}i} = 1$

$$g_{(\hat{n})}(\xi) = (\mu_i + (\hat{n}-1)\mu_{\text{-}i})e^{-\xi}e^{-(\mu_i+(\hat{n}-1)\mu_{\text{-}i})e^{-\xi}} \tag{B.39}$$

The expectation of the match-specific component of demand for $\hat{n}$ draws is denoted by $q(a, \hat{n})$,

$$\begin{aligned} q(\hat{n}) &\equiv \bar{\Gamma}^{-1}\int_{-\infty}^{\infty} e^{\sigma(\kappa-1)\xi}g_{(\hat{n})}(\xi)\mathrm{d}\xi &\text{(B.40)}\\ &= \bar{\Gamma}^{-1}\int_{-\infty}^{\infty} e^{\sigma(\kappa-1)\xi}(\mu_i + (\hat{n}-1)\mu_{\text{-}i})e^{-\xi}e^{-(\mu_i+(\hat{n}-1)\mu_{\text{-}i})e^{-\xi}}\mathrm{d}\xi &\text{(B.41)}\\ &= \bar{\Gamma}^{-1}(\mu_i + (\hat{n}-1)\mu_{\text{-}i})\int_{-\infty}^{\infty} e^{-(1-\sigma(\kappa-1))\xi}e^{-(\mu_i+(\hat{n}-1)\mu_{\text{-}i})e^{-\xi}}\mathrm{d}\xi &\text{(B.42)}\end{aligned}$$

Use the $\int_{-\infty}^{\infty} e^{-Ax}e^{-Be^{-x}}\mathrm{d}x = B^{-A}\Gamma(A)$ form, as before, and noting that $\bar{\Gamma} \equiv \Gamma(1-\sigma(\kappa-1))$

$$= (\mu_i + (\hat{n}-1)\mu_{\text{-}i})^{\sigma(\kappa-1)} \tag{B.43}$$

Evaluate at the symmetric μ, and use the definition of $\Psi(\mu, \hat{n})$

$$= \Psi(\mu, \hat{n})^{\sigma(\kappa-1)} \tag{B.44}$$

For a given distribution, age-dependent distribution of $\hat{n}$, denote the expected match specific quality as,

$$\begin{aligned} q(a) &\equiv \mathbb{E}_a\left[q(\hat{n})\right] &\text{(B.45)}\\ &= \mathbb{E}_a\left[\Psi(\mu, \hat{n})^{\sigma(\kappa-1)}\right] &\text{(B.46)}\end{aligned}$$

Note that this definition for $q(a)$ corresponds to the $\mu = \mu_{\text{-}i}$ definition in Main (15)

Industry age enters the matches through a distribution of n for the continuum of industries of a particular age. Rearranging (B.13). Convert to the $\hat{n}$ random variable and use (A.63),

$$P = \bar{\Gamma}^{-1}\left(\int (1 - f_0(a))\hat{p}(a)^{1-\kappa}\mathbb{E}_a\left[\int_{-\infty}^{\infty} e^{\sigma(\kappa-1)\xi} g_{(\hat{n})}(\xi)\mathrm{d}\xi\right] M\mathrm{d}\Phi(t,a)\right)^{\frac{1}{1-\kappa}} \tag{B.47}$$

Substitute into this using (B.46) and Main (15) to find the price index

$$P(t) = \left(\int (1 - f_0(a))\hat{p}(t,a)^{1-\kappa} q(a) M\mathrm{d}\Phi(t,a)\right)^{\frac{1}{1-\kappa}} \tag{B.48}$$

B.5 Evolution of the Age Distribution

To provide a baseline model for a stationary age distribution, assume that consumer taste shocks (potentially induced by the creation of new varieties, or due to Shumpeterian forces) make product categories obsolete and kill industries at a constant rate $\delta_M > 0$.[3] On the other side, new product categories/industries enter as a result of R&D investment at a rate $\hat{x}(t) > 0$.[4] The obsolescence rate δ_M approximates the rate at which narrowly defined product categories cease to be in consumers' choice sets.

Conditional on the product category creation rate $\hat{x}(t)$—as chosen by consumers through R&D investment i_M in Main (10)—the KFE for the normalized age distribution is

$$\boldsymbol{\partial}_t\hat{\Phi}(t,a) = \underbrace{-\boldsymbol{\partial}_a\hat{\Phi}(t,a)}_{\text{Age Increase}} - \underbrace{\delta_M\hat{\Phi}(t,a)}_{\text{Obsolescence}} + \underbrace{\hat{x}(t)}_{\substack{\text{New Prod.}\\\text{Categories}}} \tag{B.49}$$

Given the total number of product categories $M(t)$, define the proportional entry rate as $x(t) \equiv \hat{x}(t)/M(t)$ and the normalized age distribution as $\Phi(t,a) \equiv \hat{\Phi}(t,a)/M(t)$. Then, given entry rate $\hat{x}(t)$ and an initial condition $\hat{\Phi}(0,a)$, the number of product categories evolves according to

$$\boldsymbol{\partial}_t M(t) = -\delta_M M(t) + \hat{x}(t), \quad \text{s.t. } M(0) = \hat{\Phi}(0,\infty) \tag{B.50}$$

With constant entry and destruction rates, the proportional entry converges to the obsolescence rate, $x = \delta_M$; the total mass of product categories is $M = \hat{x}/\delta_M$; and the normalized, stationary distribution of product category ages is exponentially distributed,

$$\Phi(a) = 1 - e^{-\delta_M a} \tag{B.51}$$

Proof of (B.49) *to* (B.51). First, note that if a flow of $\hat{x}(t)$ industries are born and a proportional flow of δ_M are removed due to obsolescence, then the law of motion for the total mass of industries is

$$M'(t) = -\delta_M M(t) + \hat{x}(t) \tag{B.52}$$

Rearrange and use $\hat{x}(t) = x(t)M(t)$,

$$\frac{M'(t)}{M(t)} + \delta_M = x(t) \tag{B.53}$$

[3] An alternative version of the model would be a (potentially endogenous) growth model with no product category depreciation, and some decreasing returns in $\boldsymbol{\partial}_t \log M(t)$. Atkeson and Burstein (2019) provide an interpretation of "social discounting of innovation" in models with innovation of varieties and productivity as analogous to physical capital depreciation if there was no investment in innovation. If the version presented here is considered a normalization to a BGP, my model could be interpreted in a similar way.

[4] This approach emphasizes the creation and destruction of new products and varieties rather than of firms. Broda and Weinstein (2010) similarly discusses changes in the varieties available to consumers. In this paper, while firms produce only one product, at the aggregate level this distinction does not matter, and the organization of products among firms is indeterminate as long as firms have no span-of-control issues.

Take derivatives of $\Phi(t,a) \equiv \hat{\Phi}(t,a)/M(t)$,

$$\boldsymbol{\partial}_a \hat{\Phi}(t,a) = M(t)\boldsymbol{\partial}_a \Phi(t,a) \tag{B.54}$$

$$\boldsymbol{\partial}_t \hat{\Phi}(t,a) = M(t)\boldsymbol{\partial}_t \Phi(t,a) + M'(t)\Phi(t,a) \tag{B.55}$$

Substitute these expressions, $\hat{\Phi}(t,a) = M(t)\Phi(t,a)$, and $\hat{x}(t) = x(t)M(t)$ into (B.49),

$$\boldsymbol{\partial}_t \Phi(t,a) = -\boldsymbol{\partial}_a \Phi(t,a) - \left(\delta_M + \frac{M'(t)}{M(t)}\right)\hat{\Phi}(t,a) + x(t) \tag{B.56}$$

Use (B.53) and reorganize

$$\boldsymbol{\partial}_t \Phi(t,a) = -\boldsymbol{\partial}_a \Phi(t,a) + (1 - \Phi(t,a))\, x(t) \tag{B.57}$$

Also, given an $M(t)$ function, substitute from (B.53) to get

$$\boldsymbol{\partial}_t \Phi(t,a) = -\boldsymbol{\partial}_a \Phi(t,a) + (1 - \Phi(t,a))\left(\frac{\boldsymbol{\partial}_t M(t)}{M(t)} + \delta_M\right) \tag{B.58}$$

To get the stationary distribution, note from (B.53) that a $x(t) = \delta_M$ is necessary. Substitute this into (B.57) to get the ODE,

$$0 = \boldsymbol{\partial}_a \Phi(a) + \delta_M(1 - \Phi(a)) \tag{B.59}$$

Solve the ODE subject to the initial condition $\Phi(0) = 0$ and $\Phi(a) = 1$ for $a > 0$,

$$\Phi(a) = 1 - e^{-\delta_M a} \tag{B.60}$$

□

B.6 Dynamic Equilibrium

See Appendix B.5 for a derivation of the normalization and law of motion of the age distribution. Given an initial condition for the economy (i.e., $k(0)$, $M(0)$, $\Phi(0,a)$), the following characterize the transition dynamics:[5]

Proposition 7 (Dynamic Equilibrium)**.** *When $i_M(t)$ is non-binding, the equilibrium relationship between $\hat{M} \equiv M(t)/z_M(t)$ and $k(t)$ solves the following implicit equation,*

$$\delta_M - \delta_k + \boldsymbol{\partial}_t \log z_M(t) = z(t)Q(t)B(t)^{-1} z_M(t)^{\frac{1}{\kappa-1}} k^{\alpha} \hat{M}^{\frac{1}{\kappa-1}} \left(\tfrac{1}{\kappa-1}\hat{M}^{-1} - \alpha k^{-1}\right) \tag{B.61}$$

Define a function, $\zeta(\cdot)$ as the solution to this implicit equation such that $\hat{M} = \zeta(t,k)$. Then given $z(t)$ and $z_M(t)$, the following system of ODEs in $C(t)$ and $k(t)$ characterize the equilibrium

$$\boldsymbol{\partial}_t k(t) = \frac{z(t)Q(t)B(t)^{-1}k(t)^{\alpha} z_M(t)^{\frac{1}{\kappa-1}} \zeta(t,k(t))^{\frac{1}{\kappa-1}} - C(t) - \delta_k k(t) - (\delta_M + \boldsymbol{\partial}_t \log z_M(t))\zeta(t,k(t)) - \boldsymbol{\partial}_t \zeta(t,k(t))}{1 + \boldsymbol{\partial}_k \zeta(t,k(t))} \tag{B.62}$$

$$\boldsymbol{\partial}_t \log C(t) = \frac{1}{\gamma}\left(\alpha z(t)Q(t)B(t)^{-1}k(t)^{\alpha-1} z_M(t)^{\frac{1}{\kappa-1}} \zeta(t,k(t))^{\frac{1}{\kappa-1}} - \rho - \delta_k\right) \tag{B.63}$$

[5] A consideration here is the assumption of conversion of consumption goods to capital or new industries. In order to simplify the interpretation of comparative statics, I assume that the markets for creating new industries and new capital have the same awareness friction as the market for consumption goods, and the same α. If I wanted to have completely frictionless markets for homogeneous capital and variety intermediates, I could simply adjust the productivity to undo the awareness wedge for that market—e.g., $\boldsymbol{\partial}_t k(t) = -\delta_k k(t) + Z(t)Q(t)^{-1}B(t)k(t)^{\alpha} i_k(t)$.

Where $Q(t)$ and $B(t)$ are a function of $\Phi(t,a)$, and given the equilibrium $M(t)$, $\Phi(t,a)$ evolves according to,

$$\boldsymbol{\partial}_t \Phi(t,a) = -\boldsymbol{\partial}_a \Phi(t,a) + (1-\Phi(t,a))\left(\boldsymbol{\partial}_t \log z_M(t) + \frac{\boldsymbol{\partial}_t k(t)\boldsymbol{\partial}_k \zeta(t,k(t)) + \boldsymbol{\partial}_t \zeta(t,k(t))}{\zeta(t,k(t))} + \delta_M\right) \tag{B.64}$$

Proof. From Main (24) and (B.28) with inelastic labor supply normalized to 1, define a $b(t)$ to simplify the algebra,

$$b(t) \equiv \frac{\left[\mathbb{E}_t\left[(1-f_0(a))\Upsilon(a)^{1-\kappa}q(a)\right]\right]^{1/\varsigma}}{\mathbb{E}_t\left[(1-f_0(a))\Upsilon(a)^{-\kappa}q(a)\right]} = \frac{Q(t)}{B(t)} \tag{B.65}$$

$$Y(t) = z(t)b(t)k(t)^{\alpha}M(t)^{\frac{1}{\kappa-1}} \tag{B.66}$$

With this, define a present-value Hamiltonian for Main (28) as,

$$\begin{aligned}\mathcal{H} \equiv \frac{1}{1-\gamma}\left[\underbrace{\overbrace{z(t)b(t)M(t)^{\frac{1}{\kappa-1}}k(t)^{\alpha}}^{\equiv f(t,k,M)} - i_k(t) - i_M(t)}_{\equiv C(t)}\right]^{1-\gamma} \\ + \lambda_k(t)\left(-\delta_K k(t) + i_k(t)\right) + \lambda_M(t)\left(-\delta_M M(t) + z_M(t)i_M(t)\right) + \lambda_i i_M(t)\end{aligned} \tag{B.67}$$

where $\lambda_k(t), \lambda_M(t)$, and $\lambda_i(t)$ are co-state variables, with the complementary condition $\lambda_i(t)i_M(t) = 0$ and $i_M(t) \geq 0$. With the $\boldsymbol{\partial}_{i_k}\mathcal{H} = 0$ first-order necessary condition,

$$C(t)^{-\gamma} = \lambda_k(t) \tag{B.68}$$

Differentiate, divide by $\lambda_k(t)$, and reorganize,

$$\frac{\boldsymbol{\partial}_t \lambda_k(t)}{\lambda_k(t)} = -\gamma\frac{\boldsymbol{\partial}_t C(t)}{C(t)} = -\gamma \boldsymbol{\partial}_t \log C(t) \tag{B.69}$$

For the $\boldsymbol{\partial}_{i_M}\mathcal{H} = 0$ first-order necessary and complementarity conditions,

$$C(t)^{-\gamma} \geq z_M(t)\lambda_M(t) \tag{B.70}$$

$$= z_M(t)\lambda_M(t), \text{ if } i_M(t) > 0 \tag{B.71}$$

If the constraint is non-binding at t, then use (B.68) and (B.71), differentiate, and divide by $z_M(t)\lambda_M(t)$ to find

$$\frac{\boldsymbol{\partial}_t \lambda_M(t)}{\lambda_M(t)} = \frac{\boldsymbol{\partial}_t \lambda_k(t)}{\lambda_k(t)} - \frac{\boldsymbol{\partial}_t z_M(t)}{z_M(t)} \tag{B.72}$$

For the $\boldsymbol{\partial}_k\mathcal{H} = \rho\lambda_k(t) - \boldsymbol{\partial}_t\lambda_k(t)$ and $\boldsymbol{\partial}_M\mathcal{H} = \rho\lambda_M(t) - \boldsymbol{\partial}_t\lambda_M(t)$ first-order conditions for the present value Hamiltonian

$$C(t)^{-\gamma}\boldsymbol{\partial}_k f(t,k,M) - \lambda_k(t)\delta_k = \rho\lambda_k(t) - \boldsymbol{\partial}_t\lambda_k(t) \tag{B.73}$$

$$C(t)^{-\gamma}\boldsymbol{\partial}_M f(t,k,M) - \lambda_M(t)\delta_M = \rho\lambda_M(t) - \boldsymbol{\partial}_t\lambda_M(t) \tag{B.74}$$

Divide by $\lambda_k(t)$ and $\lambda_M(t)$ respectively, and then use (B.68), (B.69), (B.71) and (B.72)

$$\boldsymbol{\partial}_k f(t,k,M) = \rho + \delta_k + \gamma\boldsymbol{\partial}_t \log C(t) \tag{B.75}$$

$$z_M(t)\boldsymbol{\partial}_M f(t,k,M) = \rho + \delta_M + \boldsymbol{\partial}_t \log z_M(t) + \gamma\boldsymbol{\partial}_t \log C(t) \tag{B.76}$$

Combine (B.75) and (B.76) to get an expression between k and M—which is static if $z_M(t)$ is constant

$$z_M(t)\boldsymbol{\partial}_M f(t,k,M) - \boldsymbol{\partial}_k f(t,k,M) = \delta_M - \delta_k + \boldsymbol{\partial}_t \log z_M(t) \tag{B.77}$$

Assume monotonicity properties such as complementarity, $\boldsymbol{\partial}_{kM} f(t,k,M) > 0$, so there is a unique M that solves (B.77) for any given k and t. Define this function as $\zeta(\cdot)$ such that,

$$M(t) \equiv z_M(t)\zeta(t,k(t)) \tag{B.78}$$

Take the derivative with the chain rule,

$$\frac{\boldsymbol{\partial}_t M(t)}{z_M(t)} = \boldsymbol{\partial}_t \zeta(t,k(t)) + \boldsymbol{\partial}_t k(t)\boldsymbol{\partial}_k \zeta(t,k(t)) + \zeta(t,k(t))\boldsymbol{\partial}_t \log z_M(t) \tag{B.79}$$

With the law of motion for $M(t)$ in Main (30), use (B.78) and (B.79)

$$i_M(t) = \frac{\boldsymbol{\partial}_t M(t)}{z_M(t)} + \frac{\delta_M M(t)}{z_M(t)} \tag{B.80}$$

$$= \boldsymbol{\partial}_t \zeta(t,k(t)) + \boldsymbol{\partial}_t k(t)\boldsymbol{\partial}_k \zeta(t,k(t)) + \zeta(t,k(t))\boldsymbol{\partial}_t \log z_M(t) + \delta_M \zeta(t,k(t)) \tag{B.81}$$

From laws of motion for $k(t)$ in Main (29)

$$i_k(t) = \boldsymbol{\partial}_t k(t) + \delta_k k(t) \tag{B.82}$$

Use Main (31) and (B.81) and (B.82) to solve for $\boldsymbol{\partial}_t k(t)$,

$$\boldsymbol{\partial}_t k(t) = \frac{1}{1+\boldsymbol{\partial}_k \zeta(t,k(t))}\left(f(t,k(t),M(t)) - C(t) - \delta_k k(t) - (\delta_M + \boldsymbol{\partial}_t \log z_M(t))\zeta(t,k(t)) - \boldsymbol{\partial}_t \zeta(t, \tag{B.83}$$

From $f(\cdot)$ in (B.67) and (B.78), note that

$$f(t,k(t),M(t)) = z(t)b(t)k(t)^{\alpha} z_M(t)^{\frac{1}{\kappa-1}} \zeta(t,k(t))^{\frac{1}{\kappa-1}} \tag{B.84}$$

Substitute into the derivatives with (B.78),

$$\boldsymbol{\partial}_k f(t,k(t),M(t)) = \frac{\alpha}{k(t)} z(t)b(t)k(t)^{\alpha} M(t)^{\frac{1}{\kappa-1}} \tag{B.85}$$

$$= \frac{\alpha}{k(t)} z(t)b(t)k(t)^{\alpha} z_M(t)^{\frac{1}{\kappa-1}} \zeta(t,k(t))^{\frac{1}{\kappa-1}} \tag{B.86}$$

$$\boldsymbol{\partial}_M f(t,k(t),M(t)) = \frac{1}{(\kappa-1)M(t)} z(t)b(t)k(t)^{\alpha} M(t)^{\frac{1}{\kappa-1}} \tag{B.87}$$

$$= \frac{1}{(\kappa-1)z_M(t)\zeta(t,k(t))} z(t)b(t)k(t)^{\alpha} z_M(t)^{\frac{1}{\kappa-1}} \zeta(t,k(t))^{\frac{1}{\kappa-1}} \tag{B.88}$$

From (B.77)

$$\delta_M - \delta_k + \boldsymbol{\partial}_t \log z_M(t) = z_M(t)\boldsymbol{\partial}_M f(t,k,M) - \boldsymbol{\partial}_k f(t,k,M) \tag{B.89}$$

Take (B.77), (B.78), (B.85) and (B.87), substitute for the production function $f(\cdot)$, and define $\hat{M} \equiv M(t)/z_M(t)$ to get the implicit equation,

$$\delta_M - \delta_k + \boldsymbol{\partial}_t \log z_M(t) = z(t)b(t)z_M(t)^{\frac{1}{\kappa-1}} k(t)^{\alpha} \hat{M}^{\frac{1}{\kappa-1}} \left(\tfrac{1}{\kappa-1}\hat{M}^{-1} - \alpha k^{-1}\right) \tag{B.90}$$

Where the solution to this implicit equation defines the $\hat{M} \equiv M(t)/z_M(t) = \zeta(t, k(t))$ function. Substitute (B.86) into (B.75)

$$\boldsymbol{\partial}_t \log C(t) = \frac{1}{\gamma}\left(\frac{\alpha}{k(t)} z(t)b(t)k(t)^{\alpha} z_M(t)^{\frac{1}{\kappa-1}} \zeta(t, k(t))^{\frac{1}{\kappa-1}} - \rho - \delta_k\right) \tag{B.91}$$

Repeat (B.83) to finalize the set of 2 ODEs in $k(t)$ and $C(t)$ given the $\zeta(\cdot)$ function from above.

$$\boldsymbol{\partial}_t k(t) = \frac{z(t)b(t)k(t)^{\alpha} z_M(t)^{\frac{1}{\kappa-1}} \zeta(t, M(t))^{\frac{1}{\kappa-1}} - C(t) - \delta_k k(t) - (\delta_M + \boldsymbol{\partial}_t \log z_M(t))\zeta(t, k(t)) - \boldsymbol{\partial}_t \zeta(t, k(t))}{1 + \boldsymbol{\partial}_k \zeta(t, k(t))} \tag{B.92}$$

The solution is then an ODE in $C(t)$ and $k(t)$ given by (B.91) and (B.92) with $\zeta(\cdot)$ defined by (B.90). The evolution of the distribution comes directly from (B.49), which in turn determines the evolution of $b(t)$. Use (B.49) and substitute with $M(t) = \zeta(t, k(t))z_M(t)$

$$\boldsymbol{\partial}_t \Phi(t, a) = -\boldsymbol{\partial}_a \Phi(t, a) + (1 - \Phi(t, a))\left(\boldsymbol{\partial}_t \log z_M(t) + \frac{\boldsymbol{\partial}_t k(t) \boldsymbol{\partial}_k \zeta(t, k(t)) + \boldsymbol{\partial}_t \zeta(t, k(t))}{\zeta(t, k(t))} + \delta_M\right) \tag{B.93}$$

□

In Proposition 7, the equilibrium choices have been simplified to a system of 2 ODEs in $k(t)$ and $C(t)$, where given a particular $M(t)$ path, the corresponding $\Phi(t, a)$ evolution. Keeping track of $\Phi(t, a)$ is crucial since net-entry and expansion of $M(t)$ leads to changes in the age distribution, especially during transitions.

B.7 Stationary Equilibrium

Proof of Main Proposition 6. From Appendix B.5, the stationary age distribution is independent of the choice variables k and M. As derivatives of the $\zeta(\cdot)$ function drop out in the stationary equilibrium, it can be written in terms of M rather than $\hat{M}$ and $\zeta(\cdot)$. From Proposition 7, the stationary set of equations is,

$$\delta_M - \delta_k = zbk^{\alpha} M^{\frac{1}{\kappa-1}} \left(\frac{z_M}{(\kappa - 1)M} - \frac{\alpha}{k}\right) \tag{B.94}$$

From this $\hat{M} = \zeta(k)$ function, the equilibrium capital solves

$$\rho + \delta_k = \alpha z b k^{\alpha-1} M^{\frac{1}{\kappa-1}} \tag{B.95}$$

And given the k and M, from $i_k = \delta_k k$ and $i_M = \delta_M M/z_M$, the equilibrium C is

$$C = zbk^{\alpha} M^{\frac{1}{\kappa-1}} - \delta_k k - \delta_M M/z_M \tag{B.96}$$

□

B.8 Controlled Awareness

While the evolution of the economy given a fixed awareness process $\mathbb{Q}$ is covered in previous sections, with endogeneity, the off-equilibrium actions need to be considered. As quality heterogeneity is left out, this means that I need only to consider the value of a single agent deviating from a symmetric Nash equilibrium.

While the evolution of the economy given a fixed awareness process $\mathbb{Q}$ is covered in previous sections, with endogeneity, the off-equilibrium actions need to be considered. As quality heterogeneity is left out, this means that I need only to consider the value of a single agent deviating from a symmetric Nash equilibrium. The stationary equilibrium is similar to that of Main Proposition 6, except for an additional equilibrium condition ensuring that (θ, μ) is privately optimal; expectations are calculated conditional on this particular (θ, μ); and the productivity of creating new industries of the optimal type is $\hat{z}_M(\theta, \mu)$.

Proposition 8 (Stationary Equilibrium with Controlled Awareness)**.** *A symmetric, stationary equilibrium is a θ, μ, k, and M solving the system of implicit equations,*

$$\delta_M - \delta_k = Q(\theta,\mu)B(\theta,\mu)^{-1}k^{\alpha}M^{\frac{1}{\kappa-1}}\left(\frac{\hat{z}_M(\theta,\mu)}{\kappa-1}M^{-1} - \alpha k^{-1}\right) \tag{B.97}$$

$$\rho + \delta_k = \alpha Q(\theta,\mu)B(\theta,\mu)^{-1}M^{\frac{1}{\kappa-1}}k^{\alpha-1} \tag{B.98}$$

$$(\theta,\mu) = \arg\max_{\theta_i,\mu_i}\left\{\int_0^{\infty} e^{-(\rho+\delta_M)a}\frac{\theta_i}{\theta}\pi^*(a,\mu_i \,|\, \theta,\mu)\mathrm{d}a - \frac{d(\theta_i,\mu_i)}{N}\right\} \tag{B.99}$$

where $\hat{z}_M(\theta,\mu) \equiv z_M + d(\theta,\mu)^{-1}$, and $\Phi(a) = 1 - e^{-\delta_M a}$. Given the θ, μ, k and M,

$$C = Q(\theta,\mu)B(\theta,\mu)^{-1}M^{\frac{1}{\kappa-1}}k^{\alpha} - \delta_k k - \delta_M M/\hat{z}_M(\theta,\mu) \tag{B.100}$$

Proof. (B.97), (B.98) and (B.100) are identical to Main Proposition 6, except for conditioning on $\hat{z}_M(\theta,\mu)$ and a modified (θ,μ). See the full specification and derivation in Appendix B.8. $\pi^*(a,\mu_i \,|\, \theta,\mu)$ is the profits using firm i's optimal pricing strategy with their unilateral deviation, described in (B.103). ☐

Proof for Main Section 4.1. First, adapting Appendix A.4, note that with a single asymmetric firm, and $N-1$ symmetric firms, the probability of that firm being in an awareness set of size n is no longer distributed (central) Hypergeometric. Instead, with the model of investment in awareness distorting the relative probabilities, I model the probability following Fisher's non-central Hypergeometric distribution (i.e., an urn problem with no replacement and biased "weights").

Let the particular firm's choice be θ and the symmetric choice of the other firms be $\bar{\theta}$. Then, the relative weight is $\theta/\bar{\theta}$. From the probability mass function for Fisher's non-central Hypergeometric distribution, the probability of a successful draw of firm i with an awareness set of size n and N total firms is,

$$\frac{\binom{N-1}{n-1}\theta/\bar{\theta}}{\binom{N-1}{N} + \binom{N-1}{n-1}\theta/\bar{\theta}} = \frac{n}{N}\frac{\theta/\bar{\theta}}{1 + (\theta/\bar{\theta}-1)n/N} \approx \frac{\theta}{\bar{\theta}}\frac{n}{N}, \text{ for a large } N \text{ limit} \tag{B.101}$$

In the equilibrium with symmetric weights in Main Proposition 2, this is identical to the n/N derived in Appendix A.4. Furthermore, with a large N limit, deviations of θ_i from θ for a single firm have a negligible effect on the distribution of awareness set sizes for each consumer. Let $f_n(a|\theta)$ be the pmf of awareness set sizes given an equilibrium θ.

With (B.101) and (A.31), (A.56) and (A.80)

$$\pi(a,p_i,\theta_i,\mu_i \,|\, p,\theta,\mu) = \mu_i\Omega\left(p_i - mc\right)\frac{1-f_0(a|\theta)}{N}p_i^{-\kappa}\mathbb{E}_a\left[\hat{n}\hat{\Psi}(p_i,p,\mu_i,\mu,\hat{n})^{\sigma(\kappa-1)-1}\right] \tag{B.102}$$

Given a particular θ, the large N assumption leads to off-equilibrium changes in a particular θ having a small impact on the expectation

$$= \frac{\theta_i}{\theta}\pi(a,p_i,\theta,\mu_i \,|\, p,\theta,\mu) \tag{B.103}$$

A consequence of this, is that θ_i does not enter the off-equilibrium pricing strategy of firms, and that we can solve for the optimal price $p_i(a, \mu_i|p, \theta, \mu)$ from (A.77), where the optimal price of other firms is $p(a)$. Define the profits given the optimal pricing policy and fixed θ as

$$\pi^*(a, \mu_i \,|\, \theta, \mu) = \pi(a, p_i(a, \mu_i), \theta, \mu_i \,|\, p(a), \theta, \mu) \tag{B.104}$$

With this, given a discount rate of $\rho + \delta_M$ (where δ_M was the poisson death rate), the time 0 value of a firm with a fixed μ_i, θ_i given aggregates μ, θ is

$$v(\theta_i, \mu_i|\theta, \mu) = \int_0^\infty e^{-(\rho+\delta_M)a} \frac{\theta_i}{\theta} \pi^*(a, \mu_i \,|\, \theta, \mu) \mathrm{d}a \tag{B.105}$$

We will look for symmetric equilibrium where the optimal choice of θ_i and μ_i are θ and μ. □

B.9 Example Advertising Capital Production

Final goods required are

$$d(\theta, \mu) = \frac{\theta^{\eta_1} + (\mu - 1)^{\eta_2}}{\nu} \tag{B.106}$$

Where $\nu > 0$ is the productivity of creating advertising technology, $\eta_1 > 1$ is the decreasing returns to scale in choosing a high θ, and $\eta_2 > 1$ governs the cost of choosing a $\mu > 1$

Baseline parameters to start with are $\nu = 2.5 \times .0565, \eta_2 = BIG, \eta_1 = 2.5$

The goal is to target $\theta = 0.06$ or whatever our baseline is. A big η_2 to start with will help ensure that $\mu \approx 1$,

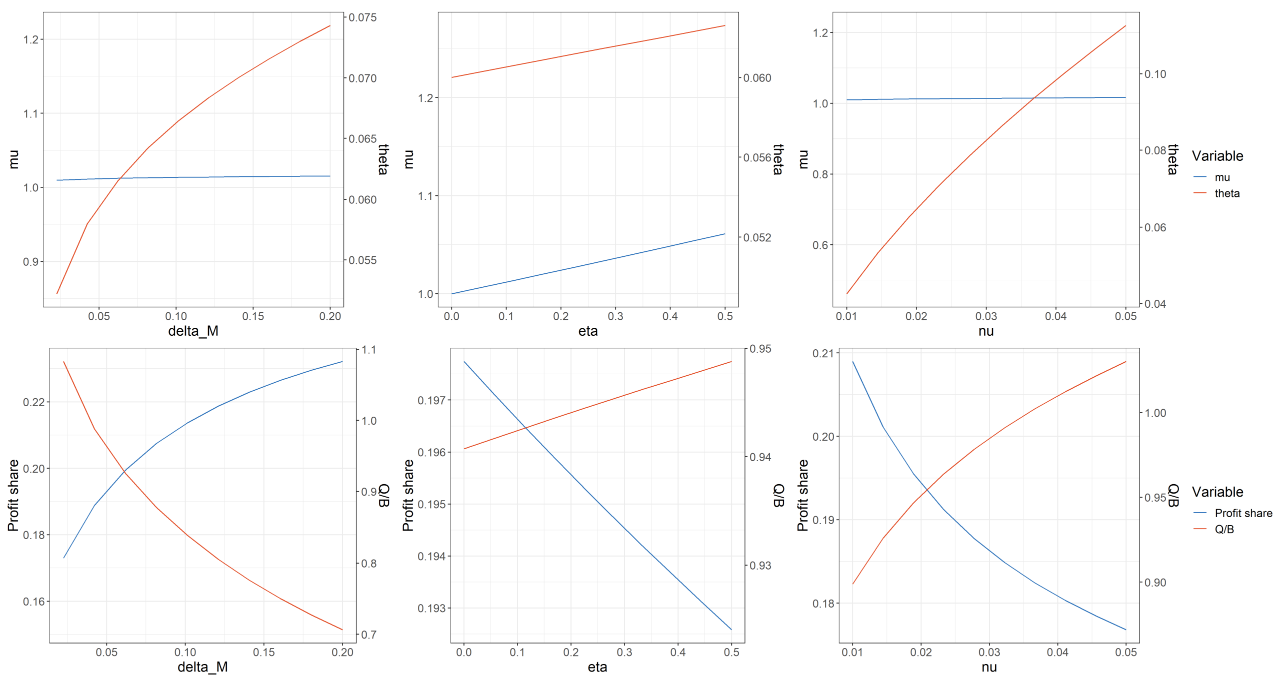

Figure 1: Comparative Statics with Endogenous Awareness under Alternative d Specification

C Multiple Cohorts

This section extends the simple examples where a single "count" was a sufficient marginal distribution for awareness set sizes. Recall that the key is to be able to calculate the demand function for each firm, and for them to be able to calculate expectations of the demand functions for other firms they may compete with (for each consumer).

C.1 Notation and Awareness Counts

To introduce multiple cohorts,

- Cohorts are indexed by $b = 1, \dots \bar{b}$ where the symmetric case in the paper is nested with $\bar{b} = 1$
- Let a_b be the industry age at birth for cohort b. Typically, $a_1 = 0$ since the industry starts with the first cohort.
- There are N_b firms in cohort b, but in the symmetric case with the same number of firms per cohort we will just use N
- Given symmetric awareness evolution within a cohort, denote the set of possible awareness set counts as $\mathcal{N}$ and the cardinality of the set of awareness sets is then

$$\mathbf{N} \equiv |\mathcal{N}| = \prod_{b=1}^{\bar{b}} (N_b + 1) \tag{C.1}$$

- In the example of cohorts of the same size, $\mathcal{N} \equiv \{0, 1, \dots N\}^{\bar{b}}$
- We can use $n_b \in \{0, \dots, N_b\}$ to be an awareness set with n_b firms in the awareness set size for cohort b. This is a generalization of the n used with a single cohort.
- The awareness state for an industry is $f(a) \in \mathbb{R}^{\mathbf{N}}$, which generalizes the one-cohort example in the main paper.
- An element from the $\mathbf{N}$ possible awareness sets is a $n \in \mathcal{N}$

$$n \equiv \{n_1, \dots n_b, \dots n_{\bar{b}}\} \tag{C.2}$$

- Denote a sum over all possible awareness count permutations as $\sum_{n \in \mathcal{N}}$, etc.

Generalizing the awareness count evolution, we now have a infinitesimal generator $Q \in \mathbb{R}^{\mathbf{N} \times \mathbf{N}}$ such that the evolution of awareness is

$$\boldsymbol{\partial}_t f(a) = f(a) \cdot Q(a, f) \tag{C.3}$$

Which nests the single-cohort case and can potentially be nonlinear or time-inhomogenous

C.2 Demand with Multiple Cohorts of Symmetric Quality

Using the notation in Appendix C.1, denote the quality and price of all forms in cohort b as q_b and p_b. For deriving the demand, we will consider firm i in cohort b as having price i, so that the demand function is

$$y_{ib}(p_i, p_{\text{-}i}, f) = \sum_{n \in \mathcal{N}} n_b f_n \left[\sum_{b'=1}^{\bar{b}} n_{b'} \left(\frac{p_{b'}}{q_{b'}} \right)^{-1/\sigma} + p_i^{-1/\sigma} - p_b^{-1/\sigma} \right]^{\sigma(\kappa-1)-1} \tag{C.4}$$

Here we drop the a since the f vector provides the sufficient information to calculate demand and prices.

To verify that the formula in (C.4) is correct, note that

$$y_{ib}(p_i, p_{\text{-}i}, f) = \bar{\Gamma}^{1-\kappa}\Omega\, q_b^{1/\sigma} p_i^{-1/\sigma-1} \sum_{A\,|\,i} \left(\hat{f}(a, A) \left[\sum_{i' \in A} \left(\frac{p_{b(i')}}{q_{b(i')}} \right)^{-1/\sigma} \right]^{\sigma(\kappa-1)-1} \right) \tag{C.5}$$

holds by (A.30) where $b(i')$ denotes the cohort firm i' belongs to.

The summation in (C.5) is an expectation with respect to the random awareness set conditional on firm i being awared. For $n \in \mathcal{N}$, note that firm i is awared if and only if A conditional on n contains firm i. Thus, (C.5) can be rewritten as

$$y_{ib}(p_i, p_{\text{-}i}, f) = \bar{\Gamma}^{1-\kappa}\Omega\, q_b^{1/\sigma} p_i^{-1/\sigma-1} \cdot \tag{C.6}$$

$$\sum_{n \in \mathcal{N}} \sum_{A\,|\,n} \left(1(i \in A) \hat{f}(a, A) \left[\sum_{i' \in A} \left(\frac{p_{b(i')}}{q_{b(i')}} \right)^{-1/\sigma} \right]^{\sigma(\kappa-1)-1} \right) \tag{C.7}$$

Since firms within the same cohort possess the same quality, fixing $n \in \mathcal{N}$ and A conditional on n, the summand within the bracket in (C.7) can be rewritten as

$$\sum_{i' \in A} \left(\frac{p_{b(i')}}{q_{b(i')}} \right)^{-1/\sigma} = \sum_{b' \neq b} n_{b'} \left(\frac{p_{b'}}{q_{b'}} \right)^{-1/\sigma} + \frac{p_i^{-1/\sigma} + (n_b - 1) p_b^{-1/\sigma}}{q_b^{-1/\sigma}} \tag{C.8}$$

$$= \sum_{b'=1}^{\bar{b}} n_{b'} \left(\frac{p_{b'}}{q_{b'}} \right)^{-1/\sigma} + \frac{p_i^{-1/\sigma} - p_b^{-1/\sigma}}{q_b^{-1/\sigma}} \tag{C.9}$$

Note that, conditional on $n_b > 0$, the probability of a b-cohort firm being included in the n_b awared firms out of total N firms is n_b/N (Polya urn). Hence, combined with (C.9), (C.5) can be written in terms of cross-cohort awareness counts as

$$y_{ib}(p_i, p_{\text{-}i}, f) = \bar{\Gamma}^{1-\kappa}\Omega q_b^{1/\sigma} p_i^{-1/\sigma-1} \frac{1}{N} \cdot \tag{C.10}$$

$$\sum_{n \in \mathcal{N}} n_b f_n \left[\sum_{b'=1}^{\bar{b}} n_{b'} \left(\frac{p_{b'}}{q_{b'}} \right)^{-1/\sigma} + \frac{p_i^{-1/\sigma} - p_b^{-1/\sigma}}{q_b^{-1/\sigma}} \right]^{\sigma(\kappa-1)-1} \tag{C.11}$$

C.3 Demand with a Single Cohort with Two Qualities

Let q_k and p_k denote the quality of all firms in quality type $k \in \{H, L\}$ where $q_L \leq q_H$. N_k represents the total number of firms with quality k. Since we consider a single cohort case, $\mathcal{N} = \{0\} \cup \mathbb{N}$. Consider the demand function faced by firm i in quality type $k \in \{H, L\}$. Note that

$$y_{ik}(p_i, p_{\text{-}i}, f) = \bar{\Gamma}^{1-\kappa}\Omega\, q_k^{1/\sigma} p_i^{-1/\sigma-1} \sum_{A\,|\,i} \left(\hat{f}(a, A) \left[\sum_{i' \in A} \left(\frac{p_{k(i')}}{q_{k(i')}} \right)^{-1/\sigma} \right]^{\sigma(\kappa-1)-1} \right) \tag{C.12}$$

holds by (A.30) where $k(i')$ denotes the quality type firm i' belongs to.

The summation in (C.12) is an expectation with respect to the random awareness set conditional on firm i being awared. For $n \in \mathcal{N}$, note that firm i is awared if and only if A conditional on n

contains firm i. Letting n_k denote the number of awared firms with k quality type, (C.12) can be rewritten as

$$y_{ik}(p_i, p_{\text{-}i}, f) = \bar{\Gamma}^{1-\kappa}\Omega\, q_k^{1/\sigma} p_i^{-1/\sigma-1} \cdot \tag{C.13}$$

$$\sum_{n\in\mathcal{N}} \sum_{A\,|\,n} \left(1(i \in A)\hat{f}(a, A) \left[\sum_{i'\in A} \left(\frac{p_{k(i')}}{q_{k(i')}} \right)^{-1/\sigma} \right]^{\sigma(\kappa-1)-1} \right) \tag{C.14}$$

$$= \bar{\Gamma}^{1-\kappa}\Omega\, q_k^{1/\sigma} p_i^{-1/\sigma-1} \cdot \tag{C.15}$$

$$\sum_{n\in\mathcal{N}} \sum_{n'_k=1}^{n} \sum_{A\,|\,n; n_k=n'_k} \left(1(i \in A)\hat{f}(a, A) \left[\sum_{i'\in A} \left(\frac{p_{k(i')}}{q_{k(i')}} \right)^{-1/\sigma} \right]^{\sigma(\kappa-1)-1} \right) \tag{C.16}$$

Since firms within the same quality group possess the same quality, fixing $n \in \mathcal{N}$ and $n_k \leq n$, by considering A conditional on n, the summand within the bracket in (C.16) can be rewritten as

$$\sum_{i'\in A} \left(\frac{p_{k(i')}}{q_{k(i')}} \right)^{-1/\sigma} = \sum_{k'\in\{H,L\}} n_{k'} \left(\frac{p_{k'}}{q_{k'}} \right)^{-1/\sigma} + \frac{p_i^{-1/\sigma} - p_k^{-1/\sigma}}{q_k^{-1/\sigma}} \tag{C.17}$$

$$= n_k \left(\frac{p_k}{q_k} \right)^{-1/\sigma} + (n - n_k) \left(\frac{p_{-k}}{q_{-K}} \right)^{-1/\sigma} + \frac{p_i^{-1/\sigma} - p_k^{-1/\sigma}}{q_k^{-1/\sigma}} \tag{C.18}$$

where subscript $-k$ denotes the other quality type different from k (H if $k = L$ and L if $k = H$). The second equation (C.18) follows from the fact that $n = n_H + n_L$.

For a given awareness set A, firm i is awared if and only if n_k firms of type k are awared and i is included in the awared n_k firms of type k, with the probability represented by

$$P(\{\text{Firm } i \text{ is awared}\} \cap \{n_k \text{ firms of type } k \text{ are awared}\}) \tag{C.19}$$

Note that (C.19) is identical with the product of $P(\{n_k$ firms of type k are awared$\})$ and $P(\{\text{Firm } i \text{ is awared}\} \mid \{n$ Conditional on n_k, the number of firms of type k awared out of n firms, the probability of firm i being included in the awared n_k firms out of total N_k firms in type k is n_k/N_k (Polya urn). Note that the probability of having n_k firms from type k out of n draws is $\binom{N_k}{n_k}\binom{N_{-k}}{n-n_k}/\binom{N_H+N_L}{n}$ (hypergeometric distribution). Hence, (C.19) is identical with

$$\frac{n_k}{N_k} \frac{\binom{N_k}{n_k}\binom{N_{-k}}{n-n_k}}{\binom{N_H+N_L}{n}} \tag{C.20}$$

Thus, combined with (C.18) and (C.20), (C.16) can be rewritten as

$$y_{ik}(p_i, p_{\text{-}i}, f) = \bar{\Gamma}^{1-\kappa}\Omega\, q_k^{1/\sigma} p_i^{-1/\sigma-1} \frac{1}{N_k} \cdot$$

$$\sum_{n\in\mathcal{N}} f_n \sum_{n_k=1}^{n} n_k \frac{\binom{N_k}{n_k}\binom{N_{-k}}{n-n_k}}{\binom{N_H+N_L}{n}} \left(n_k \left(\frac{p_k}{q_k} \right)^{-1/\sigma} + (n - n_k) \left(\frac{p_{-k}}{q_{-k}} \right)^{-1/\sigma} + \frac{p_i^{-1/\sigma} - p_k^{-1/\sigma}}{q_k^{-1/\sigma}} \right)^{\sigma(\kappa-1)-1} \tag{C.21}$$

C.3.1 Deriving Demand with Symmetric Quality from Two Quality Demand

(C.21) nests the symmetric quality formula with a single cohort, by taking $N_{-k} = 0$ and $\bar{b} = 1$. To see this, first note that

$$\frac{\binom{N_k}{n_k}\binom{0}{n-n_k}}{\binom{N_k}{n}} = \begin{cases} 1 & \text{if } n = n_H \\ 0 & \text{otherwise} \end{cases} \tag{C.22}$$

as the hypergeometric distribution has support on $\{\max(0, n+N_k-(N_H+N_L)), ..., \min(n, N_k)\}$, which is $\{n\}$ when $N_{-k}=0$. Hence, in this case, (C.21) can be rewritten as

$$y_{ik}(p_i, p_{\text{-}i}, f) = \bar{\Gamma}^{1-\kappa}\Omega\, q_k^{1/\sigma} p_i^{-1/\sigma-1}\, \frac{1}{N_k}\cdot \tag{C.23}$$

$$\sum_{n\in\mathcal{N}, n_k=n} f_n n_k \left(n_k \left(\frac{p_k}{q_k}\right)^{-1/\sigma} + (n-n_k)\left(\frac{p_{-k}}{q_{-k}}\right)^{-1/\sigma} + \frac{p_i^{-1/\sigma} - p_k^{-1/\sigma}}{q_k^{-1/\sigma}} \right)^{\sigma(\kappa-1)-1} \tag{C.24}$$

$$= \bar{\Gamma}^{1-\kappa}\Omega\, q_k^{1/\sigma} p_i^{-1/\sigma-1}\, \frac{1}{N_k}\cdot \sum_{n\in\mathcal{N}} f_n n \left(n\left(\frac{p_k}{q_k}\right)^{-1/\sigma} + \frac{p_i^{-1/\sigma} - p_k^{-1/\sigma}}{q_k^{-1/\sigma}} \right)^{\sigma(\kappa-1)-1} \tag{C.25}$$

which is equivalent with (C.10) – (C.11) by taking $N = N_k$, $p_b = p_k$, and $q_b = q_k$.

C.3.2 Extension: Demand with a Single Cohort with Multiple Qualities

The formula (C.21) can be extended to multiple $\bar{k}$ quality types with arbitrary $\bar{k} > 1$. Consider the demand function faced by firm i in quality type k as having price p_i. Using a similar argument as above, it can be shown that the demand function can be represented by

$$y_{ik}(p_i, p_{\text{-}i}, f) = \bar{\Gamma}^{1-\kappa}\Omega\, q_k^{1/\sigma} p_i^{-1/\sigma-1}\cdot \tag{C.26}$$

$$\sum_{n\in\mathcal{N}} \sum_{n'_k=1}^{n} \sum_{A\,|\,n; n_k=n'_k} \left(1(i\in A)\hat{f}(a, A) \left[\sum_{i'\in A} \left(\frac{p_{k(i')}}{q_{k(i')}}\right)^{-1/\sigma} \right]^{\sigma(\kappa-1)-1} \right) \tag{C.27}$$

We introduce $v_{n_k}(n)$, a $\bar{k}$-length vector of zeros and (or) natural numbers whose sum is n and kth element is fixed by $n_k \le n$, such that its k'th element represents the number of awared firms with type k'. Let $\mathcal{V}_{n_k}(n)$ denote the collection of all $v_{n_k}(n)$. Then, given $v \in \mathcal{V}_{n_k}(n)$, the summand in (C.27) can be represented by

$$\sum_{i'\in A} \left(\frac{p_{k(i')}}{q_{k(i')}}\right)^{-1/\sigma} = \sum_{k'=1}^{\bar{k}} v_{k'} \left(\frac{p_{k'}}{q_{k'}}\right)^{-1/\sigma} + \frac{p_i^{-1/\sigma} - p_k^{-1/\sigma}}{q_k^{-1/\sigma}} \tag{C.28}$$

where $v_{k'}$ is the k'th element of v. It is worth mentioning that the second equation in (C.18) does not follow since fixing n_k is not sufficient to figure out $n_{k'}$ for $k \neq k'$ when $\bar{k} > 2$.

For a given awareness set A, firm i of type k is awared if and only if v firms are awared for some $v \in \mathcal{V}_{n_k}(n)$ and i is included in the awared v firms, with the probability represented by

$$P(\{\text{Firm } i \text{ is awared}\} \cap \{v \text{ firms are awared}\}) \tag{C.29}$$

Note that (C.29) is identical with the product of two probabilities $P(\{v \text{ firms are awared}\})$ and $P(\{\text{Firm } i \text{ is awared}\} \mid \{v \text{ firms are awared}\})$. Conditional on $v \in \mathcal{V}_{n_k}(n)$, whose kth element is n_k (the number of firms of type k awared out of n firms), the probability of firm i being included in the awared n_k firms out of total N_k firms in type k is n_k/N_k (Polya urn). Note that the probability of having $v \in \mathcal{V}_{n_k}(n)$ firms awared out of n draws is $\prod_{k'=1}^{\bar{k}} \binom{N_{k'}}{v_{k'}} / \binom{N}{n}$ (multivariate hypergeometric distribution) where $N = \sum_{k'=1}^{\bar{k}} N_{k'}$. Hence, (C.29) is identical with

$$\frac{n_k}{N_k} \frac{\prod_{k'=1}^{\bar{k}} \binom{N_{k'}}{v_{k'}}}{\binom{N}{n}} \tag{C.30}$$

Thus, combined with (C.28) and (C.30), we have

$$y_{ik}(p_i, p_{\text{-}i}, f) = \bar{\Gamma}^{1-\kappa}\Omega\, q_k^{1/\sigma} p_i^{-1/\sigma-1}\, \frac{1}{N_k}\cdot \tag{C.31}$$

$$\sum_{n\in\mathcal{N}} f_n \sum_{n_k=1}^{n} n_k \sum_{v\in\mathcal{V}_{n_k}(n)} \frac{\prod_{k'=1}^{\bar{k}}\binom{N_{k'}}{v_{k'}}}{\binom{N}{n}} \left(\sum_{k'=1}^{K} v_{k'} \left(\frac{p_{k'}}{q_{k'}}\right)^{-1/\sigma} + \frac{p_i^{-1/\sigma} - p_k^{-1/\sigma}}{q_k^{-1/\sigma}}\right)^{\sigma(\kappa-1)-1} \tag{C.32}$$

C.4 Demand with Multiple Cohorts with Two Qualities

Consider firm i of type $k \in \{H, L\}$ in cohort b as having price p_{ikb}; the demand function firms in the other type, L, can be deduced symmetrically. Without loss of generality, we assume that N_H and N_L are fixed for all cohorts for brevity.

Let $\mathcal{V}(n)$ denote the collection of $\bar{b}$-length vectors whose b'th element represents the number of awared firms of type k from cohort b' given n, i.e., vectors whose b'th element $n_{kb'}$ has the value of $\{0, ..., n_{b'}\}$. It is worth mentioning that given n, the size of $\mathcal{V}(n)$ is $\prod_{b'=1}^{\bar{b}}(n_{b'}+1)$. Then, the demand function can be represented by

$$y_{ikb}(p_i, p_{\text{-}i}, f) = \bar{\Gamma}^{1-\kappa}\Omega\, q_k^{1/\sigma} p_i^{-1/\sigma-1}\cdot \tag{C.33}$$

$$\sum_{n\in\mathcal{N}} \sum_{n'_k=1}^{n} \sum_{A\,|\,n;n_k=n'_k} \left(1(i\in A)\hat{f}(a, A)\left[\sum_{i'\in A}\left(\frac{p_{k(i')}}{q_{k(i')}}\right)^{-1/\sigma}\right]^{\sigma(\kappa-1)-1}\right) \tag{C.34}$$

holds by (A.30) where $b(i')$ denotes the cohort firm i' belongs to.

Let $\mathcal{V}(n)$ denote the collection of $\bar{b}$-length vectors whose b'th element represents the number of awared firms of type k from cohort b' given n, i.e., vectors whose b'th element $n_{kb'}$ has the value of $\{0, ..., n_{b'}\}$. It is worth mentioning that given n, the size of $\mathcal{V}(n)$ is $\prod_{b'=1}^{\bar{b}}(n_{b'}+1)$. Then, as in (C.28), the demand function can be represented as

$$\begin{aligned} y_{ikb}(p_i, p_{\text{-}i}, f) &= \bar{\Gamma}^{1-\kappa}\Omega\, q_k^{1/\sigma} p_i^{-1/\sigma-1}\, \frac{1}{N_k}\cdot \\ &\sum_{n\in\mathcal{N}} f_n \sum_{n_k\in\mathcal{V}(n)} \Bigg\{ P(\{\text{Firm } i \text{ is awared}\} \cap \{n_k \text{ firms are awared}\})\cdot \\ &\left[\sum_{b'=1}^{\bar{b}}\left(n_{kb'}\left(\frac{p_{kb'}}{q_k}\right)^{-1/\sigma} + (n-n_{kb'})\left(\frac{p_{(-k)b'}}{q_{-k}}\right)^{-1/\sigma}\right) + \frac{p_i^{-1/\sigma} - p_{kb}^{-1/\sigma}}{q_k^{-1/\sigma}}\right]^{\sigma(\kappa-1)-1} \Bigg\} \end{aligned} \tag{C.35}$$

Note that $P(\{\text{Firm } i \text{ is awared}\} \cap \{n_k \text{ firms are awared}\})$ in (C.35) is identical with the product of the following two probabilities: $P(\{\text{Firm } i \text{ is awared} \mid n_k \text{ firms are awared}\})$ and $P(\{n_k \text{ firms are awared}\})$. Conditional on $n_k \in \mathcal{V}(n)$ and n, the number of firms of type k awared out of n firms, the probability of firm i being included in the awared n_{kb} firms out of total N_k firms in type k is n_{kb}/N_k (Polya urn). As in (C.21), for each b' cohort, the probability of having $n_{kb'}$ firms from type k out of $n_{b'}$ draws is $\binom{N_k}{n_{kb'}}\binom{N_{-k}}{n_{b'}-n_{kb'}}/\binom{N_H+N_L}{n_{b'}}$ (hypergeometric distribution). Since the awareness process is independent across all cohorts, $v \in \mathcal{V}(n)$ firms awared out of n draws is $\prod_{b'=1}^{\bar{b}}\left[\binom{N_k}{n_{kb'}}\binom{N_{-k}}{n_{b'}-n_{kb'}}/\binom{N_H+N_L}{n_{b'}}\right]$. Hence, the demand function is

$$
\begin{aligned}
&y_{ikb}(p_i, p_{\text{-}i}, f) = \bar{\Gamma}^{1-\kappa}\Omega\, q_k^{1/\sigma} p_i^{-1/\sigma-1}\, \frac{1}{N_k}\cdot \\
&\sum_{n\in\mathcal{N}} f_n \sum_{n_k\in\mathcal{V}(n)} \Bigg\{ \left[n_{kb} \prod_{b'=1}^{\bar{b}} \frac{\binom{N_k}{n_{kb'}}\binom{N_{-k}}{n_{b'}-n_{kb'}}}{\binom{N_H+N_L}{n_{b'}}} \right] \cdot \\
&\left[\sum_{b'=1}^{\bar{b}} \left(n_{kb'} \left(\frac{p_{kb'}}{q_k}\right)^{-1/\sigma} + (n - n_{kb'}) \left(\frac{p_{(-k)b'}}{q_{-k}}\right)^{-1/\sigma} \right) + \frac{p_i^{-1/\sigma} - p_{kb}^{-1/\sigma}}{q_k^{-1/\sigma}} \right]^{\sigma(\kappa-1)-1} \Bigg\}
\end{aligned}
\tag{C.36}
$$

D Asymmetric Firms

This section gives a few examples of asymmetry to clarify the empirical predictions for different intrinsic quality and/or entry timing. The asymmetric version of demand and prices is nested in the derivation of Appendix A.[6]

To demonstrate the forces and qualitative results, the numerical examples use a duopoly. The theory, however, fully extends to an arbitrary number of firms (or types of firms). Remember that, in practice, awareness needs to be tracked only for firms with a distinct quality and entry date, which simplifies the evolution of awareness and the computational burden.

These examples are also consistent with the explanation in Syverson (2004) that the degree of productivity dispersion that can be sustained in an industry is a function of the degree of substitutability between the products.

D.1 Asymmetric Entry

Empirical studies such as Foster et al. (2016) find evidence that entrants have systematically different prices and productivity than incumbents, but grow very slowly—even when the physical TFP and intrinsic quality of the incumbent and entrant are identical.

Assuming that the process for awareness set sizes, $\hat{n}$, is a sub-martingale, prices (and markups) are weakly decreasing over time. At first, the fact that younger firms tend to have higher prices would seem to contradict the micro-evidence. For example, Foster et al. (2016) finds that new entrants temporarily have only slightly higher productivity, but they tend to price significantly lower than incumbents before both prices and productivity converge. However, recall that the symmetric solutions requires that *both* symmetric intrinsic quality and symmetric time of entry. In this model, if a firm with identical intrinsic quality entered later, consumers would have asymmetric awareness sets.

This asymmetry in awareness makes the pricing decision dependent on firm age, as shown in Figure 2. For an incumbent, decreasing the price would decrease profits from existing customers who are unaware of the entrant. In contrast, the entrant has monopoly power over very few of its customers and, in the BNE, can lower prices to temporarily attract the incumbent's customers. Moreover, the average match quality of a consumer—conditional on purchasing the product—is different between the incumbent and the entrant. Incumbents sell to a number of consumers who would otherwise choose the entrant if it were in the consumer's choice set, while the entrant needs to compete for most of its consumers. This leads to an asymmetric degree of sorting, and,

[6] To summarize the changes to the evolution of the distributions: with asymmetry, I need to add in a placeholder for those different types of firms into the awareness states, and expand the size of $\mathbb{Q}$. To deal with future entry, I can make the $\mathbb{Q}$ matrix time varying and/or add in placeholders for future entry types. Either way, there is no harm in having a placeholder awareness state which is of measure 0 until the entry occurs. The $f(a)$ for a given $\mathbb{Q}$ is calculated numerically, and efficiently, as a system of ODEs. Hence, even if $\mathbb{Q}$ is time varying and has a cardinality of tens of thousands, a solution is numerically tenable.

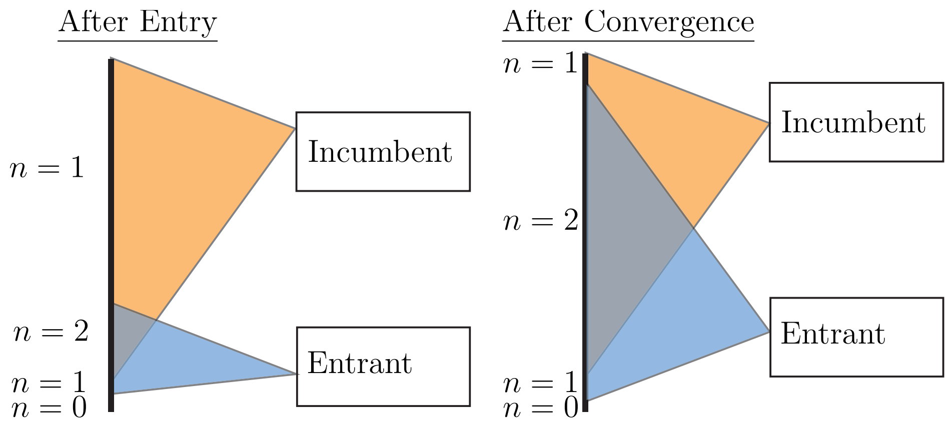

Figure 2: Asymmetric Entry Leads to Asymmetric Market Power and Sorting

consequently, better average match quality for entrants, and it would show as higher revenue TFP even if absolute demand is low. Over time, as the incumbent loses customers due to more direct price competition with the incumbent, the retained customers will tend to have a higher average match quality, leading to converging revenue TFP as the incumbent's market share decreases. If the awareness process $\mathbb{Q}$ is ergodic, the prices and quality levels converge as the awareness sets become similar—as the micro-evidence suggests.

This suggests caution in interpreting increases in quality or TFP after an increase in the number of competitors as evidence of competition spurring quality innovation—especially if market shares decrease post-entry or after splitting a monopoly. Here, the increase in quality and aggregate output of the industry after entry is purely a passive process of consumer sorting. Nevertheless, even if the intrinsic properties of the products have not changed, the increase in average quality from sorting is real from a consumer's perspective.

Numerical Example To see this effect, the following experiment solves the calibrated model with 80% of the population initially aware of firm 1, a monopolist—i.e., $\int_{[0,1]} \mathbb{1}\{1 \in A_j(0)\}\,\mathrm{d}j = 0.8$. At that point, firm 2 enters with identical quality and no initial awareness—i.e., $\int_{[0,1]} \mathbb{1}\{2 \in A_j(0)\}\,\mathrm{d}j = 0$.

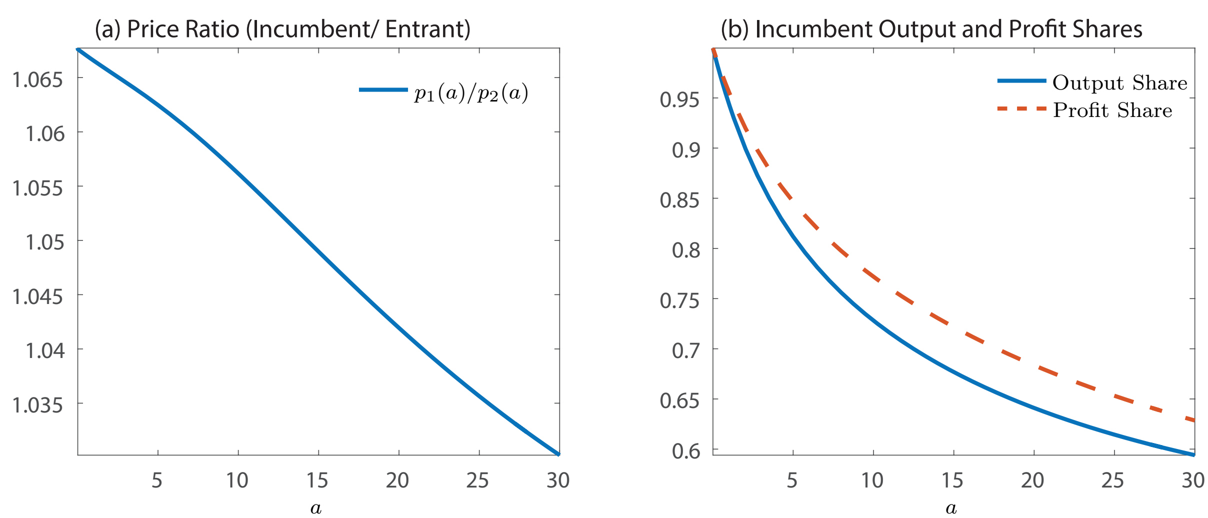

Figure 3: Entry into a Monopoly

The effect of asymmetry on pricing is evident in panel (a) of Figure 3, where the entrant has weakly lower prices throughout the industry life cycle. To understand the incentives in the BNE game: Initially, the incumbent charges the standard constant markup over marginal price inherent in monopolistic competition, and captured in the κ elasticity of substitution (and independent of the dispersion of idiosyncratic preferences within the product category, σ). As discussed, the key is that the incumbent does not want to lose monopoly profits from the 80% of consumers who start with awareness of the incumbent, but no awareness of the entrant. While the entrant could charge the identical price as the incumbent, it has the incentive to lower its prices and capture a fraction of consumers who would otherwise choose the incumbent if prices were identical. How much the entrant is able to exploit this asymmetry in information sets is inherent in the dispersion of idiosyncratic preferences, σ. For the incumbent, lowering the price decreases monopoly profits from existing consumers without full awareness, which prevents the incumbent from retaliating by also lowering its prices.

During the transition towards symmetric information sets, note that the output and profit shares of the incumbent in panel (b) of Figure 3 are skewed. In particular, the previous monopolist is able to capture a greater proportion of industry profits than the entrant due to its ability to exploit the market power inherent in limited choice sets. Following through with this logic by considering standard model extensions: (1) this shows that if a firm is able to be a first mover in a market, there are positive profits inherent in the slow growth of firms and information frictions, even without further incentives, such as intellectual property protection. So, contrary to many endogenous growth models, policies to ensure the protection of market power (e.g., the patent system) are not necessary to ensure positive profits for the creation of a new industry; and (2) if fixed costs were added to the model through standard mechanisms, the asymmetry in the fierceness of competition for consumers could either drive entrants out of business, or simply deter entry. On the other hand, if there are complementaries in the evolution of awareness (such as high probabilities that consumers will find other providers once they have at least one product in their awareness set for the product category), there could be incentives to let other firms build initial awareness of the product category before entry.

The convergence of the prices in Figure 3 shows that as awareness sets become more similar, this effect disappears and prices converge towards the symmetric equilibrium of Main Proposition 3. Hence, driven only by the information asymmetry in awareness, entrants have higher quality (or, equivalently, revenue productivity) and lower prices than incumbents, but these differences disappear over time—consistent with Foster et al. (2016). Beyond urging caution in interpreting evidence of entry spurring intrinsic changes in incumbents, this highlights that firms are symmetric only if they have both identical production technologies and customer awareness.

D.2 Asymmetric Quality (or Productivity)

Many industries have a different skewness of market share versus profit share. For example, Apple took 91% of smartphone profits in 2015, while having only 17.2% of the market share—all in a competitive and fairly mature industry. This model shows that small differences in the intrinsic quality of products can lead to larger differences in profit-share than market-share, and that the differences in quality (and the different skews) would seem more pronounced as the industry evolved. Furthermore, Syverson (2004) shows enormous and sustained variations in profitability and productivity can be sustained within an industry, and this example shows how this is possible in my model.

With asymmetry in underlying quality, the key force in this model is that with a high dispersion of preferences within the product category, σ, sorting of consumers to preferred products can have a strong effect on profit shares. However, sorting takes time to develop, as it requires choice sets with more than a single firm. Consequently, as an industry matures and consumers become aware of all of the competitors, the profit share becomes much more skewed.

Numerical Example In order to understand the dynamics of market and profit shares, the following experiment of the calibrated model takes a duopoly with asymmetric firm quality. Both firms enter at age $a = 0$, but one firm has a 10% higher average quality than the other (i.e., $q_h/q_\ell = 1.1$ in the notation of Appendix A). As always, in this model, quality and productivity differences would be isomorphic given only revenue and profit data.

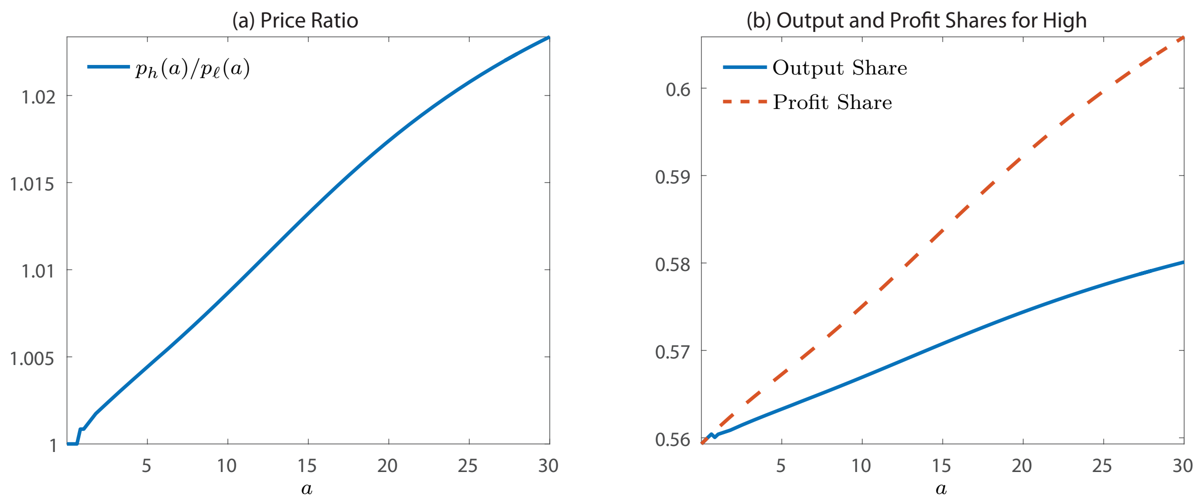

Figure 4: Evolution of High vs. Low Quality

In panel (a) of Figure 4, the higher-quality firm has slightly higher prices per unit of the good (reflecting the higher average-quality consumer's gain), but the price difference is asymptotically very small. The prices start out as identical because both firms are able to charge the monopolistically competitive price. It is only as the industry matures that the extent of the higher quality becomes evident, when consumers have several goods in their choice sets.

In contrast, panel (b) of Figure 4 shows that the profit and output-share become much more skewed. The output and profit shares are identical at the beginning since most consumers have only a single good in their choice set, but starts slightly higher than 50/50 since consumers have higher intensive demand for the higher quality product. This diverges as consumers sort into their preferred product, and the higher quality good can be priced accordingly to maximize profits given their position. The asymmetry between the profit ratio and customer ratio explains how a few firms in an industry can capture the majority of the industry's profits, but not necessarily capture the same proportion of customers or output.[7]

E Parameters "Calibration" Summary

As the model is left with the minimum number of parameters, any simulations should be considered more qualitative than quantitative. However, I have calibrate the parameters by connecting theoretical elements of the model to industry panel-data from the NBER-CES Manufacturing Industry Database.

[7] To gain a profit share split as skewed as the Apple example, we would need to combine asymmetric quality with the first-mover entry in Appendix D.1 (which also leads to a more skew), and probably add in fixed operating costs to both amplify the profit shares dispersion and show why many phone producers have negative profits.

	mean	sd	min	max
Minimum Price Cost Markup	0.21	0.09	-0.19	0.50
Maximum Price Cost Markup	0.39	0.11	0.18	1.30
Observations	4406			

Table 1: Summary Statistics for Min and Max Markup by Industry

Data Sources The benefit of using the NBER-CES Manufacturing Industry Database (MID) is that: (1) It has very precise available data on markups at the industry level, without the selection issues of large firms that could be caused by using Compustat; (2) it tracks output of the industry as a whole from relatively early age, which makes it more appropriate than something like Compustat with severe selection issues and difficulty determining the age of an industry, or aggregating output.

On the other hand, directly using the panel is smoothing out many of the dispersion in growth rates, likely leading to slower evolution of awareness than a panel of firms. Hence the resulting θ and θ_d may be too low

Markups Table 1 provides a summary of the average highest and the average lowest markups in the MID industry panel. Using the asymptotic theory of markups from, these numbers provide bounds on σ and κ. In particular, the minimum markup is bounded by $1+\sigma$, and the maximum markup is bounded by the monopolistically competitive $\kappa/(\kappa-1)-1$.

Industry Growth Rates The parameters θ and θ_d are calibrated from the MID while examining at the growth rate of industry output, and are very industry dependent. The approach is to use nonlinear least squares to match the proportion of industry output, as a function of the peak industry output age, compared to the solution of the markov chain using θ and θ_d. Since these parameters are so industry dependent, these values should only be considered a rough approximation for qualitative purposes.

Product Obsolescence In the simple aggregate model with a constant hazard, the survivor function for product categories is, $\text{Survival}(a) = e^{-\delta_M a}$
The obsolescence rate δ_M is difficult to measure directly since it is related to product category—*not* firm—exit rates. I consider several alternatives to discipline the parameter:

- Reinterpreting Example 2 in Atkeson and Burstein (2019) as a measure of obsolescence, provides a δ_M of 0.0225.
- Broda and Weinstein (2010) finds that households spend 20% of their money on goods that will disappear in the next 4 years. Using the 80% survival after 4 years leads to $\delta_M = 0.056$.
- Finally, using trademark data directly from my own calculations, if the survival rate of trademarks is 16% after 10 years, then $\delta_M = 0.18$.

As the baseline value for calibrations, I will use the 0.056 estimate since it is the only direct measure of products (calculated from Nielson scanner data) and is in the middle of the range.

[8] Recall that due to the distortions in markups and profits, this can no longer be calibrated directly from the labor share proportion.

Variable	Value	Description
σ	≤ 0.21	Uses the theoretical minimum industry markup bound from Proposition 5—i.e., $p(a)/mc \geq 1+\sigma$. Markups are calculated as the *average minimum markup* from NBER-CES Manufacturing Industry Database following Nekarda and Ramey (2011). Baseline is $\sigma = 0.15$.
$\kappa/(\kappa-1)-1$	≥ 0.39	Use the theoretical maximum industry bound from Proposition 5—i.e., $p(a)/mc \leq \kappa/(\kappa-1)$ Markups are calculated as the *average maximum markup* from NBER-CES Manufacturing Industry Database. Baseline is $\kappa = 3.5$
θ	0.06	From Nonlinear Least Squares and industry panel growth rates with the NBER-CES Manufacturing Industry Database, fitting the S-curve shape of industry output as a function of age
θ_d	0.21	rom Nonlinear Least Squares and industry panel growth rates with the NBER-CES Manufacturing Industry Database, fitting the S-curve shape of industry output as a function of age.
δ_M	$[0.0225, 0.18]$	From Broda and Weinstein (2010), trademark obsolescence rates, or reinterpreting Atkeson and Burstein (2019). Uses $\delta_M = 0.056$ from Broda and Weinstein (2010) as the baseline.
N	N/A if > 10	With the θ and θ_d above, the N is essentially irrelevant (as long as it is above 5-10). Growth in $\hat{n}$ is too slow to converge close to N prior to obsolescence.
δ_K	0.07	Typical capital depreciation rate
α	0.28	Set from the 1980 corporate labor share in the data, with the factor share distortion, $B(t)$, derived in Main Section 3.[8]
ρ	0.03	Typical interest rate target
γ	$[1, 5]$	Typical range of elasticity of intertemporal substitution. Does not enter in steady-state comparative statics
ν	0.0178923	Targets the θ baseline from above when endogenous
z, z_M	N/A	Level effects, not calibrated

Table 2: Rough Example Parameter Calibration

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