Equilibrium Imitation and Growth
Online Technical Appendix

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Appendix A Solving for the BGP Equilibrium

References with no prefix refer to this Technical Appendix, while those with a prefix of *PT.* refer to equations in the main paper.

A.1 The Firm Problem

This version of the firm problem includes taxes and subsidies and is featured in the constrained planner problem found in Section *PT*.4. When taxes and subsidies are zero, i.e. $\tau = \varsigma = 0$, this becomes the firm problem developed in Section *PT*.2.3 and throughout the rest of the main paper.

The following extends the firm's problem in equation *PT*.2 with a constant proportional tax on production, $0 \leq \tau < 1$, and a constant proportional search subsidy, $0 \leq \varsigma < (1-\tau)$. Although searchers do not produce, they may receive positive period profits from the search subsidy, depending on the value of ς.[1]

$$V_t(z) = \max\left\{(1-\tau)z + \tfrac{1}{1+r_t}V_{t+1}(z), \quad \varsigma z + \tfrac{1}{1+r_t}\int_{m_{t+1}}^{\infty} V_{t+1}(z')\tfrac{f_t(z')}{1-F_t(m_{t+1})}\mathrm{d}z'\right\} \tag{A.1}$$

Define the *gross* value of search, before any costs/subsidies, at time t as

$$W_t \equiv \tfrac{1}{1+r_t}\int_{m_{t+1}}^{\infty} V_{t+1}(z')\frac{f_t(z')}{1-F_t(m_{t+1})}\mathrm{d}z' \tag{A.2}$$

Define the *net* value of search as $\varsigma z + W_t$.

A.2 Guesses to be Verified

Make the following guesses, which will be verified:

1. The Pareto distribution will fulfill the BGP conditions, including Scale Invariance as described in Section *PT*.2.4.2. Given an initial pdf, $f_0(z; m_0, \alpha) = \alpha m_0^{\alpha} z^{-\alpha-1}$, the distribution will evolve according to the truncation law of motion in equation *PT*.4. Given this law of motion and initial distribution, $f_t(z) = \alpha m_t^{\alpha} z^{-\alpha-1}$.

2. The reservation productivity will grow geometrically: $m_{t+1} = g m_t$.

[1] Another interpretation of ς is that only a fraction, $(1-\varsigma)$, of production is necessary to pay the cost of search. Additionally, the model can be solved with proportional costs rather than subsidies, $-(1-\tau) < \varsigma < 0$, if an external frictionless market is assumed to exist for firms to finance these additional search costs.

3. The *gross* value of search grows geometrically from some constant W. Hence, the *net* value of search is affine in m_t (or linear if $\varsigma = 0$).

$$V_t(z) = m_t W + \varsigma z, \quad \text{for } m_t \le z \le g m_t \tag{A.3}$$

No guesses or assumptions are made on the structure or linearity of $V_t(z)$ for $z > g m_t$.[2] The solution methodology, following the search literature, solves for firms optimal policies without needing to solve for V_t.

It is straightforward to prove that the Pareto distribution fulfills the BGP equilibrium requirements, such as Scale Invariance in equation $PT.6$, for any m_0. Moreover, given $m_{t+1} = g m_t$, it can be shown that $Y_{t+1} = g Y_t$ for all m_t.

To verify the last 2 guesses, it suffices to solve for constants g and W that are not a function of m_t.

A.3 Detailed Algebra for BGP Proof

Given that $Y_{t+1} = g Y_t$, the interest rate is constant: $r = \frac{g^\gamma}{\beta} - 1$. Inserting the Pareto $f_t(\cdot)$ and $m_{t+1} = g m_t$ into equation A.1 to obtain

$$\begin{aligned} V_t(z) = \max \Big\{ &(1-\tau) z + \tfrac{1}{1+r} V_{t+1}(z), \\ &\varsigma z + \tfrac{1}{1+r} \alpha (g m_t)^\alpha \int_{g m_t}^{\infty} V_{t+1}(z') z'^{-\alpha-1} \mathrm{d}z' \Big\} \end{aligned} \tag{A.4}$$

Note that with the guess that $m_{t+1} = g m_t$, the indifference level of productivity at time t is $g m_t$. Thus,

$$V_t(g m_t) = (1-\tau) g m_t + \tfrac{1}{1+r} V_{t+1}(g m_t) \tag{A.5}$$

$$= \varsigma g m_t + \tfrac{1}{1+r} \alpha (g m_t)^\alpha \int_{g m_t}^{\infty} V_{t+1}(z') z'^{-\alpha-1} \mathrm{d}z' \tag{A.6}$$

Using the guess on the affine value of search from equation A.3 with equations A.5 and A.6 gives two equalities:

$$m_t W + \varsigma g m_t = (1-\tau) g m_t + \tfrac{1}{1+r} \left(g m_t W + \varsigma g m_t \right) \tag{A.7}$$

$$(1-\tau) g m_t + \tfrac{1}{1+r} \left(g m_t W + \varsigma g m_t \right) = \varsigma g m_t + \tfrac{1}{1+r} \alpha (g m_t)^\alpha \int_{g m_t}^{\infty} V_{t+1}(z') z'^{-\alpha-1} \mathrm{d}z' \tag{A.8}$$

Use equation A.7 to obtain one equation in W and g

$$W = \frac{\left(1 - \tau - \frac{r}{1+r}\varsigma\right) g}{1 - g/(1+r)} \tag{A.9}$$

Note that m_t has dropped out of the equation, which is part of the verification that W and g are constants that are independent of the scale of the economy. Rearrange equation A.8 and split the integral at the indifference point for $t+1$:

$$\begin{aligned} (1 - \tau - \tfrac{r}{1+r}\varsigma) g m_t + \tfrac{1}{1+r} g m_t W = &\tfrac{1}{1+r} \alpha (g m_t)^\alpha \int_{g m_t}^{g^2 m_t} V_{t+1}(z') z'^{-\alpha-1} \mathrm{d}z' \\ &+ \tfrac{1}{1+r} \alpha (g m_t)^\alpha \int_{g^2 m_t}^{\infty} V_{t+1}(z') z'^{-\alpha-1} \mathrm{d}z' \end{aligned} \tag{A.10}$$

[2] In fact, $V_t(z)$ will always be non-linear due to the z dependent option value of future search. Section B uses the solution to the firm problem to solve for the value function explicitly. Equation B.7 provides an equation and economic interpretation for $V_t(z)$.

By the decision rule, firms will search at $t+1$ if $z' \leq g^2 m_t$ with value $gm_tW + \varsigma z'$. Thus,

$$\begin{aligned}
\int_{gm_t}^{g^2 m_t} V_{t+1}(z') z'^{-\alpha-1} \mathrm{d}z' &= \int_{gm_t}^{g^2 m_t} \left(gm_tW + \varsigma z'\right) z'^{-\alpha-1} \mathrm{d}z' \\
&= \frac{\varsigma gm_t}{\alpha-1}(gm_t)^{-\alpha}(1-g^{1-\alpha}) + \frac{gm_tW}{\alpha}(gm_t)^{-\alpha}(1-g^{-\alpha}) \\
&= gm_t(gm_t)^{-\alpha}\left(\frac{\varsigma}{\alpha-1}(1-g^{1-\alpha}) + \frac{W}{\alpha}(1-g^{-\alpha})\right) \qquad \text{(A.11)}
\end{aligned}$$

By the decision rule, firms will produce at $t+1$ if $z' > g^2 m_t$. Thus,

$$\begin{aligned}
\int_{g^2 m_t}^{\infty} V_{t+1}(z') z'^{-\alpha-1} \mathrm{d}z' &= \int_{g^2 m_t}^{\infty} \left((1-\tau)z' + \tfrac{1}{1+r} V_{t+2}(z')\right) z'^{-\alpha-1} \mathrm{d}z' \qquad \text{(A.12)} \\
&= \tfrac{1}{\alpha-1}(1-\tau)(g^2 m_t)^{1-\alpha} + \tfrac{1}{1+r}\int_{g^2 m_t}^{\infty} V_{t+2}(z') z'^{-\alpha-1} \mathrm{d}z' \qquad \text{(A.13)}
\end{aligned}$$

Using the indifference equation at $t+1$, where the reservation productivity is $g^2 m_t$, yields

$$\begin{aligned}
V_{t+1}(g^2 m_t) &= (1-\tau)g^2 m_t + \tfrac{1}{1+r}\left(g^2 m_t W + \varsigma g^2 m_t\right) \qquad \text{(A.14)} \\
&= \varsigma g^2 m_t + \tfrac{1}{1+r}\alpha (g^2 m_t)^{\alpha}\int_{g^2 m_t}^{\infty} V_{t+2}(z') z'^{-\alpha-1} \mathrm{d}z' \qquad \text{(A.15)}
\end{aligned}$$

Using the equality between equations A.14 and A.15 and rearranging for the integral yields

$$\tfrac{1}{1+r}\alpha(gm_t)^{\alpha}\int_{g^2 m_t}^{\infty} V_{t+2}(z') z'^{-\alpha-1} \mathrm{d}z' = g^{-\alpha} g^2 m_t \left(1 - \tau - \tfrac{r}{1+r}\varsigma + \tfrac{1}{1+r}W\right) \qquad \text{(A.16)}$$

Note that equation A.16 gives the second part of the integral in equation A.13. Combining equations A.11, A.13, and A.16 with equation A.10 provides an equation independent of value functions:

$$\begin{aligned}
(1-\tau-\tfrac{r}{1+r}\varsigma)gm_t + \tfrac{1}{1+r}gm_tW &= \tfrac{1}{1+r}\alpha(gm_t)^{\alpha} gm_t (gm_t)^{-\alpha}\left(\tfrac{\varsigma}{\alpha-1}(1-g^{1-\alpha}) + \tfrac{W}{\alpha}(1-g^{-\alpha})\right) \\
&+ \tfrac{1}{1+r}\alpha(gm_t)^{\alpha}\tfrac{1}{\alpha-1}(1-\tau)(g^2 m_t)^{1-\alpha} \\
&+ \tfrac{1}{1+r} g^{-\alpha} g^2 m_t \left(1-\tau-\tfrac{r}{1+r}\varsigma + \tfrac{1}{1+r}W\right) \qquad \text{(A.17)}
\end{aligned}$$

Dividing by gm_t and simplifying shows that

$$\begin{aligned}
1-\tau-\tfrac{r}{1+r}\varsigma + \tfrac{1}{1+r}W &= \tfrac{1}{1+r}\tfrac{\alpha}{\alpha-1}\varsigma - \tfrac{1}{1+r}\tfrac{\alpha}{\alpha-1}\varsigma g g^{-\alpha} \\
&+ \tfrac{1}{1+r}W - \tfrac{1}{1+r}W g^{-\alpha} \\
&+ \tfrac{1}{1+r}\tfrac{\alpha}{\alpha-1}(1-\tau) g g^{-\alpha} \\
&+ \tfrac{1}{1+r} g g^{-\alpha}\left(1-\tau-\tfrac{r}{1+r}\varsigma\right) + \tfrac{1}{1+r} g^{-\alpha}\tfrac{g}{1+r}W \qquad \text{(A.18)}
\end{aligned}$$

Multiplying by $g^{\alpha}(1+r)$ and rearranging yields a second equation in W and g:

$$\left((1-\tau)(1+r) - \varsigma\left(r + \tfrac{\alpha}{\alpha-1}\right)\right) g^{\alpha} = g\left(1-\tau+(1-\tau)\tfrac{\alpha}{\alpha-1} - \varsigma\tfrac{\alpha}{\alpha-1} - \varsigma\tfrac{r}{r+1}\right) - \left(1 - g/(1+r)\right) W \qquad \text{(A.19)}$$

Substituting for W from equation A.9 and simplifying gives

$$\left((1-\tau)(1+r) - \varsigma\left(r + \tfrac{\alpha}{\alpha-1}\right)\right) g^{\alpha} = g\tfrac{\alpha}{\alpha-1}\left(1-\tau-\varsigma\right) \qquad \text{(A.20)}$$

Solving for g, we have shown that

$$g = \left[\left(\frac{1-\tau-\varsigma}{1-\tau-\varsigma\left(\frac{r}{1+r}+\frac{1}{1+r}\frac{\alpha}{\alpha-1}\right)} \right) \frac{1}{1+r}\frac{\alpha}{\alpha-1} \right]^{\frac{1}{\alpha-1}} \tag{A.21}$$

As m_t has dropped out of equations A.21 and A.9, the guesses in Section A.2 of the functional form $V_t(z) = Wm_t + \varsigma z$ for $m_t \leq z \leq gm_t$ and $m_{t+1} = gm_t$ for constant W and g are verified. Note that W and g are not functions of time or the minimum of support of f_0. Intuitively, this means that the growth rate is independent of the initial scale of the economy, m_0, and inductively it is independent of the scale of the economy for any t since $m_t = m_0 g^t$. Equation A.21 can be compared to the solution without taxes or subsidies presented in equation $PT.18$.

To solve for g entirely in terms of model intrinsics, the growth rate needs to be solved as a system of equations with the interest rate given by

$$\frac{1}{1+r} = \beta g^{-\gamma} \tag{A.22}$$

Direct substitution of this interest rate into equation A.21 yields an implicit expression for g in terms of model parameters. For a general α, this implicit equation doesn't appear to always have an explicit analytical formula for g, but it can be solved explicitly if $\varsigma = 0$ or $\gamma = 0$. Note that proportional taxes do not distort growth rates in the absence of subsidies. For $\varsigma = 0$, as shown in Proposition $PT.1$,

$$g = \left(\beta \frac{\alpha}{\alpha-1}\right)^{\frac{1}{\gamma-1+\alpha}} \tag{A.23}$$

Parameter constraints are needed to ensure that $g > 1$ and $W > 0$, as can be seen in equations A.9 and A.21. Given an equilibrium r, the following are sufficient[3]

1. $\varsigma + \tau < 1$
2. $1-\tau-\varsigma\left(\frac{r}{1+r}+\frac{1}{1+r}\frac{\alpha}{\alpha-1}\right) > 0$
3. $\frac{1}{1+r}\frac{\alpha}{\alpha-1} > 1$

Appendix B Solving for the Value Function

The firm's optimal policy is to search at time t if and only if its idiosyncratic productivity is below the reservation productivity threshold m_{t+1}. Define the number of periods a firm with productivity z at time t waits before searching as:

$$\xi_t(z) \equiv \arg\min_{s\in\mathbb{N}} \{z \leq m_{t+1+s}\} \tag{B.1}$$

For example, if a firm at time t has productivity $z \leq m_{t+1}$, then the firm searches immediately and $\xi_t(z) = 0$. With a productivity of $m_{t+1} < z \leq m_{t+2}$ then the firm waits one period, etc. If, for some s, the firm has productivity $z = m_{t+1+s}$, then the firm is indifferent between waiting for s and $s+1$ periods before searching.

Recall the gross value of search at period t (before any costs/subsidies) is

$$W_t = \tfrac{1}{1+r_t} \int_{m_{t+1}}^{\infty} V_{t+1}(z') \frac{f_t(z')}{1-F_t(m_{t+1})} \mathrm{d}z' \tag{B.2}$$

In sequential space, given a Competitive Equilibrium—F_0 and $\{m_t, V_t(\cdot), r_t\}$—and the corresponding $\{W_t, \xi_t(\cdot)\}$, the value function of a firm with productivity z at time t is the discounted sum of production until search plus the net value of search at time $t+\xi_t(z)$.[4] Noting that $W_{t+\xi_t(z)}$

[3] For the specific case of $\varsigma = 0$, Proposition $PT.1$ gives, in closed form, the necessary and sufficient parameter constraints in terms of intrinsics.

[4] The convention used is that for $b < a$, $\sum_a^b = 0$ and $\prod_a^b = 1$.

already contains the discount term $\frac{1}{1+r_{t+\xi_t(z)}}$,

$$V_t(z) = (1-\tau)z \sum_{s=0}^{\xi_t(z)-1} \left(\prod_{i=0}^{s-1} \frac{1}{1+r_{t+i}}\right) + \varsigma z \left(\prod_{i=0}^{\xi_t(z)-1} \frac{1}{1+r_{t+i}}\right) + \left(\prod_{i=0}^{\xi_t(z)-1} \frac{1}{1+r_{t+i}}\right) W_{t+\xi_t(z)} \tag{B.3}$$

If the interest rate is constant (e.g., if $\gamma = 0$ or if the economy is on a BGP), then the expression can be simplified further

$$V_t(z) = \left[(1-\tau)\tfrac{1+r}{r}\left(1 - \left(\tfrac{1}{1+r}\right)^{\xi_t(z)}\right) + \varsigma \left(\tfrac{1}{1+r}\right)^{\xi_t(z)}\right] z + \left(\tfrac{1}{1+r}\right)^{\xi_t(z)} W_{t+\xi_t(z)} \tag{B.4}$$

$$= \underbrace{(1-\tau)\tfrac{1+r}{r} z}_{\text{Value of production}} + \underbrace{\left(\tfrac{1}{1+r}\right)^{\xi_t(z)} \left[\varsigma z + W_{t+\xi_t(z)} - (1-\tau)\tfrac{1+r}{r} z\right]}_{\text{Option value of search}} \tag{B.5}$$

At time t, this function is piecewise linear with kinks at $\{m_s\}$ for all $s \geq t$. Where $m_s < z < m_{s+1}$, the slope of the value function in z is the present discounted value of post-tax production for $\xi_t(z) - 1$ periods plus the value of the search subsidy discounted $\xi_t(z)$ periods. In equation B.5, this is interpreted as the value of production in perpetuity plus the option value of search. The option value of search is the value of receiving, at $\xi_t(z)$ periods in the future, the subsidy and the expected value of a new productivity draw minus the lost value of producing with z in perpetuity after the $\xi_t(z)$ periods.

It can be shown that for a given t, the option value of search is decreasing in z and asymptotically 0, since for large z the search option is executed far in the future. From equation B.5, as $\xi \to \infty$, $\left(\frac{1}{1+r}\right)^{\xi_t(z)} \to 0$. Hence, as long as $W_{t+\xi_t(z)}$ does not grow too fast, the option value of search goes to 0 as the waiting time goes to infinity.[5] This condition in the balanced growth path will be fulfilled if in equilibrium $1 + r > g$. Therefore, from B.5, the value function is approximately linear and independent of t for very large z relative to the current minimum of support m_t

$$V_t(z) \approx (1-\tau)\tfrac{1+r}{r} z, \text{ for } z \gg m_t \tag{B.6}$$

Equation B.4 can be simplified further along the balanced growth path, along which $m_t = m_0 g^t$ and $W_t = m_0 g^t W$. Substituting, the value function on the BGP is defined piecewise $\forall s \in \mathbb{N}$ as

$$V_t(z) = \begin{cases} \left[(1-\tau)\tfrac{1+r}{r}\left(1 - \left(\tfrac{1}{1+r}\right)^{s}\right) + \varsigma \left(\tfrac{1}{1+r}\right)^{s}\right] z + \left(\tfrac{1}{1+r}\right)^{s} m_0 g^{t+s} W & \text{for } z \in [m_0 g^{t+s}, m_0 g^{t+s+1}] \\ \varsigma z + m_0 g^t W & \text{for } z \leq m_0 g^t \end{cases} \tag{B.7}$$

Appendix C Normalization and Stationarity

A normalized version of the problem can be solved numerically for arbitrary initial conditions.[6]

C.1 Normalization Definitions

Given an initial condition $f_0(z) \equiv f(z)$ and the optimal reservation productivities that characterize optimal firm policies, $\{m_{t+1}\}$, define the following:

[5] Or equivalently, as z goes to infinity, since $\xi_t(z)$ is an increasing function.

[6] This stationary transformation of the model can also be used to numerically solve for model variations where the law of motion is not a simple truncation. The algorithm and code described in Appendix E are intended to support these sorts of extensions.

Normalized productivity level:

$$\tilde{z} \equiv \tfrac{z}{m_t}, \quad \text{for } \tilde{z} \in [1, \max \operatorname{support}\{f\}/m_t] \tag{C.1}$$

The support follows from the definition $m_t \equiv \min \operatorname{support}\{f_t\}$ since the law of motion ensures $\max \operatorname{support}\{f_t\} = \max \operatorname{support}\{f\}$.

Growth factor of the minimum of support:

$$g_t \equiv \tfrac{m_{t+1}}{m_t} \tag{C.2}$$

Normalized value function:

$$V_t(z) \equiv m_t \tilde{V}_t(\tfrac{z}{m_t}) \tag{C.3}$$

Normalized pdf:

$$f_t(z) \equiv \tfrac{1}{m_t} \tilde{f}_t(\tfrac{z}{m_t}) \tag{C.4}$$

Integrating the normalized pdf from m_t to z gives the normalized cdf:

$$F_t(z) \equiv \tilde{F}_t(\tfrac{z}{m_t}) \tag{C.5}$$

Rearranging the normalization in equation C.4 and using $z = \tilde{z}m_t$, gives an equivalent transformation

$$\tilde{f}_t(\tilde{z}) = m_t f_t(\tilde{z}m_t) \tag{C.6}$$

C.2 Normalized Law of Motion

Together, the law of motion from equation $PT.4$ and equation C.6 generate

$$f_t(\tilde{z}m_t) = \frac{f(\tilde{z}m_t)}{1 - F(m_t)} \tag{C.7}$$

$$\tilde{f}_t(\tilde{z}) = \frac{m_t f(\tilde{z}m_t)}{1 - F(m_t)} \tag{C.8}$$

Integrating to find the cdf yields

$$\tilde{F}_t(\tilde{z}) = \int_1^{\tilde{z}} \frac{f(\tilde{z}' m_t)}{1 - F(m_t)} m_t \mathrm{d}\tilde{z}' \tag{C.9}$$

Doing a change of variables for $z = \tilde{z}m_t$ gives

$$\tilde{F}_t(\tilde{z}) = \frac{F(\tilde{z}m_t) - F(m_t)}{1 - F(m_t)} \tag{C.10}$$

Using the law of motion, $f_{t+1}(z) = \frac{f_t(z)}{1 - F_t(m_{t+1})}$, use $z = \tilde{z}m_{t+1}$ and substitute with equation C.4 and C.5 to obtain

$$\tilde{f}_{t+1}(\tilde{z}) = m_{t+1} \frac{\frac{1}{m_t}\tilde{f}_t(\frac{z}{m_t}\frac{m_{t+1}}{m_{t+1}})}{1 - \tilde{F}_t(m_{t+1}/m_t)} = \frac{g_t \tilde{f}_t(\tilde{z}g_t)}{1 - \tilde{F}_t(g_t)} \tag{C.11}$$

Thus, the law of motion in the normalized $\tilde{z}$ space is entirely determined by the initial condition, $\tilde{f}_0(\tilde{z})$, and the sequence of growth factors, $\{g_t\}$.

Using equation C.8, define the asymptotic normalized distribution as

$$\tilde{f}_\infty(\tilde{z}) \equiv \lim_{t\to\infty} \frac{m_t f(\tilde{z}m_t)}{1-F(m_t)} \tag{C.12}$$

Also, define the asymptotic growth factor of the minimum of support as

$$g_\infty \equiv \lim_{t\to\infty} g_t \tag{C.13}$$

As a check that this normalization is stationary for the balanced growth path, use a Pareto initial condition—$f(z) = \alpha m_0^\alpha z^{-\alpha-1}$—and ensure it is constant and equal to the normalized Pareto distribution for all t:

$$\tilde{f}_t(\tilde{z}) = m_t \frac{\alpha m_0^\alpha \left(\tilde{z}m_t\right)^{-\alpha-1}}{1-\left(1-\left(\frac{m_0}{m_t}\right)^\alpha\right)} \tag{C.14}$$

$$= \alpha \tilde{z}^{-\alpha-1},\ \tilde{z}\in[1,\infty)\,\forall\, t \tag{C.15}$$

$$= \tilde{f}_\infty(\tilde{z}) = \tilde{f}_0(\tilde{z}) \tag{C.16}$$

C.3 Normalized Value Functions

Using equation A.1 and substituting the normalized versions of each variable and function gives

$$m_t\tilde{V}_t(\tfrac{z}{m_t}) = \max\Big\{(1-\tau)z + \tfrac{1}{1+r_t} m_{t+1}\tilde{V}_{t+1}(\tfrac{z}{m_{t+1}}),$$
$$\varsigma z + \tfrac{1}{1+r_t}\frac{1}{1-\tilde{F}_t\left(\frac{m_{t+1}}{m_t}\right)}\int_{m_{t+1}}^\infty m_{t+1}\tilde{V}_{t+1}\left(\tfrac{z'}{m_{t+1}}\right)\tfrac{1}{m_t}\tilde{f}_t\left(\tfrac{z'}{m_t}\right)\mathrm{d}z'\Big\} \tag{C.17}$$

Dividing by m_t and using $\tilde{z} = \frac{z}{m_t}$ yields

$$\tilde{V}_t(\tilde{z}) = \max\Big\{(1-\tau)\tilde{z} + \tfrac{1}{1+r_t}\tfrac{m_{t+1}}{m_t}\tilde{V}_{t+1}\left(\tilde{z}\tfrac{m_t}{m_{t+1}}\right),$$
$$\varsigma\tilde{z} + \tfrac{1}{1+r_t}\frac{m_{t+1}/m_t}{1-\tilde{F}_t\left(\frac{m_{t+1}}{m_t}\right)}\int_{m_{t+1}}^\infty \tilde{V}_{t+1}\left(\tfrac{z'}{m_{t+1}}\right)\tfrac{1}{m_t}\tilde{f}_t\left(\tfrac{z'}{m_t}\right)\mathrm{d}z'\Big\} \tag{C.18}$$

Using the change of variables formula $\int_a^b m(n(q))n'(q)\mathrm{d}q = \int_{n(a)}^{n(b)} m(s)\mathrm{d}s$ yields[7]

$$\tilde{V}_t(\tilde{z}) = \max\left\{(1-\tau)\tilde{z} + \tfrac{1}{1+r_t} g_t\tilde{V}_{t+1}(\tilde{z}/g_t), \varsigma\tilde{z} + \tfrac{1}{1+r_t} g_t \tfrac{1}{1-\tilde{F}_t(g_t)}\int_{g_t}^\infty \tilde{V}_{t+1}(\tilde{z}'/g_t)\tilde{f}_t(\tilde{z}')\mathrm{d}\tilde{z}'\right\} \tag{C.19}$$

The indifference point, g_t, is the root of the following equation

$$0 = (1-\tau)g_t + \tfrac{1}{1+r_t} g_t\tilde{V}_{t+1}(1) - \left(\varsigma g_t + \tfrac{1}{1+r_t} g_t\tfrac{1}{1-\tilde{F}_t(g_t)}\int_{g_t}^\infty \tilde{V}_{t+1}(\tilde{z}'/g_t)\tilde{f}_t(\tilde{z}')\mathrm{d}\tilde{z}'\right) \tag{C.20}$$

If the environment is stationary, then the problem can be written recursively. In that case, there should exist a $g, \tilde{V}(\cdot)$, and $1/(1+r) = \beta g^{-\gamma}$ that solve the following fixed point problem

$$\tilde{V}(\tilde{z}) = \max\left\{(1-\tau)\tilde{z} + \tfrac{1}{1+r} g\tilde{V}(\tilde{z}/g), \varsigma\tilde{z} + \tfrac{1}{1+r} g\tfrac{1}{1-\tilde{F}(g)}\int_g^\infty \tilde{V}(\tilde{z}'/g)\tilde{f}(\tilde{z}')\mathrm{d}\tilde{z}'\right\} \tag{C.21}$$

[7]For this change of variables: $q = z'$, $n(\cdot) = \frac{\cdot}{m_t}$, $m(\cdot) = \tilde{V}_{t+1}\left(\cdot\frac{m_t}{m_{t+1}}\right)\tilde{f}(\cdot)$.

Some systems may become stationary asymptotically or may be stationary after a change of variables.

To find the normalized version of the sequence space formulation in equation B.4, define the normalization of the gross value of search relative to time t as

$$\tilde{W}_{s|t} \equiv \frac{W_{t+s}}{m_t} \tag{C.22}$$

Normalize the optimal waiting time until search in equation B.1 such that

$$\begin{aligned}\tilde{\xi}_t(\tilde{z}) &= \arg\min_{s\in\mathbb{N}} \{\tilde{z}m_t \le m_{t+1+s}\} \\ &= \arg\min_{s\in\mathbb{N}} \{\tilde{z} \le m_{t+1+s}/m_t\}\end{aligned} \tag{C.23}$$

Note that the argmin is the same after this change of variables: $\tilde{\xi}_t(\tilde{z}) = \xi_t(z)$.

Since $g_t \equiv m_{t+1}/m_t$, m_t can be constructed from the sequence of g_t as

$$m_t = m_0 \prod_{t'=0}^{t-1} g_{t'},\ t \ge 1 \tag{C.24}$$

Define the normalized future indifference points relative to the minimum of support at t as

$$\tilde{m}_{s|t} \equiv \prod_{t'=0}^{s} g_{t+t'},\ \forall s \ge 0, t \ge 0 \tag{C.25}$$

Note that $\{\tilde{m}_{s|t} | s \ge 0\}$ are the locations in the $\tilde{z}$ domain of the kinks in the normalized value function at time t (e.g. $\{g_t, g_t g_{t+1}, \ldots\}$) and

$$\begin{aligned}\tilde{\xi}_t(\tilde{z}) &= \arg\min_{s\in\mathbb{N}} \left\{ \tilde{z} \le \frac{m_0 \prod_{t'=0}^{t+s} g_{t'}}{m_0 \prod_{t'=0}^{t-1} g_{t'}} \right\} \\ &= \arg\min_{s\in\mathbb{N}} \{\tilde{z} \le \tilde{m}_{s|t}\}\end{aligned} \tag{C.26}$$

If the interest rate is constant, equation B.4 can be used to derive the normalized value function. Dividing by m_t, and using $\tilde{z} = z/m_t$, the normalized value function is

$$\tilde{V}_t(\tilde{z}) = \left[(1-\tau)\tfrac{1+r}{r}\left(1 - \left(\tfrac{1}{1+r}\right)^{\tilde{\xi}_t(\tilde{z})}\right) + \varsigma\left(\tfrac{1}{1+r}\right)^{\tilde{\xi}_t(\tilde{z})}\right]\tilde{z} + \left(\tfrac{1}{1+r}\right)^{\tilde{\xi}_t(\tilde{z})} \tilde{W}_{\tilde{\xi}_t(\tilde{z})|t} \tag{C.27}$$

This further reduces for $\tilde{z}$ at future normalized indifference points to

$$\tilde{V}_t(\tilde{m}_{t+1+s|t}) = \left[(1-\tau)\tfrac{1+r}{r}\left(1 - \left(\tfrac{1}{1+r}\right)^{s}\right) + \varsigma\left(\tfrac{1}{1+r}\right)^{s}\right]\tilde{m}_{t+1+s|t} + \left(\tfrac{1}{1+r}\right)^{s} \tilde{W}_{s|t} \tag{C.28}$$

Given $\{g_t, W_t\}$, equations C.22, C.26, and C.27 combine to deliver the normalized value function. Along the balanced growth path, g_t is constant. Thus,

$$\tilde{W}_{s|t} = \frac{m_t g^s W}{m_t} = g^s W \tag{C.29}$$

$$\tilde{W}_{0|t} = W,\ \forall\, t \tag{C.30}$$

$$\tilde{\xi}_t(\tilde{z}) = \arg\min_{s\in\mathbb{N}} \{\tilde{z} \le g^{s+1}\} \tag{C.31}$$

$$\tilde{V}_t(\tilde{z}) = \left[(1-\tau)\tfrac{1+r}{r}\left(1 - \left(\tfrac{1}{1+r}\right)^{\tilde{\xi}_t(\tilde{z})}\right) + \varsigma\left(\tfrac{1}{1+r}\right)^{\tilde{\xi}_t(\tilde{z})}\right]\tilde{z} + \left(\tfrac{1}{1+r}\right)^{\tilde{\xi}_t(\tilde{z})} g^{\tilde{\xi}_t(\tilde{z})} W \tag{C.32}$$

Since along the balanced growth path these functions are all independent of t, the normalized value function is constant on the BGP, given by

$$\tilde{V}(\tilde{z}) = \left[(1-\tau)\tfrac{1+r}{r}\left(1 - \left(\tfrac{1}{1+r}\right)^{s}\right) + \varsigma\left(\tfrac{1}{1+r}\right)^{s}\right]\tilde{z} + \left(\tfrac{1}{1+r}\right)^{s} g^s W,\ \tilde{z} \in [g^s, g^{s+1}] \tag{C.33}$$

In particular, at the indifference points, the value of a firm is

$$\tilde{V}(g^s) = \left[(1-\tau)\tfrac{1+r}{r}\left(1 - \left(\tfrac{1}{1+r}\right)^{s}\right) + \varsigma\left(\tfrac{1}{1+r}\right)^{s}\right] g^s + \left(\tfrac{1}{1+r}\right)^{s} g^s W,\ \text{for } s \ge 0 \tag{C.34}$$

C.4 Normalized Production and Interest Rate

Production in period t is

$$Y_t = \int_{m_{t+1}}^{\infty} z f_t(z)\mathrm{d}z \tag{C.35}$$

Substituting in the normalized distribution and reorganizing shows

$$Y_t = m_t \int_{m_{t+1}}^{\infty} \tfrac{z}{m_t} \tilde{f}_t(z/m_t) \tfrac{1}{m_t}\mathrm{d}z \tag{C.36}$$

Doing a change of variables in the integral yields

$$Y_t = m_t \int_{g_t}^{\infty} \tilde{z}\tilde{f}_t(\tilde{z})\mathrm{d}\tilde{z} \tag{C.37}$$

From the consumer's optimization problem, the interest rate satisfies

$$\begin{aligned} \tfrac{1}{1+r_t} &= \beta\left(\tfrac{Y_{t+1}}{Y_t}\right)^{-\gamma} \\ &= \beta g_t^{-\gamma}\left(\frac{\int_{g_{t+1}}^{\infty} \tilde{z}\tilde{f}_{t+1}(\tilde{z})\mathrm{d}\tilde{z}}{\int_{g_t}^{\infty} \tilde{z}\tilde{f}_t(\tilde{z})\mathrm{d}\tilde{z}}\right)^{-\gamma} \end{aligned} \tag{C.38}$$

The g_t, defined here as the growth factor of the minimum of support, may not be the growth factor of production off the BGP. Along the BGP, where $\tilde{f}_t$ is stationary and g_t is constant, this yields the BGP interest rate

$$r = \tfrac{g^{\gamma}}{\beta} - 1 \tag{C.39}$$

C.5 Asymptotic Growth in Equilibrium

Proof of Proposition PT.2.

To show that power laws contradict the condition in Proposition PT.2 that $\lim_{m\to\infty} \frac{\mathbb{E}[z|z>m]}{m} = 1$, assume that $F_0(z)$ is a power law with tail parameter $\alpha > 1$. Using the normalizations defined in Section C.2, $\frac{\mathbb{E}[z|z>m_t]}{m_t} = \int_1^{\infty} \tilde{z}\frac{m_t f_0(m_t\tilde{z})}{1-F_0(m_t)}\mathrm{d}\tilde{z}$. Simple calculations show that $\lim_{m\to\infty} \frac{\mathbb{E}[z|z>m]}{m} = \frac{\alpha}{\alpha-1} > 1$. To show that an initial distribution that is a power law generates an asymptotic BGP in which the growth rate is a function of the tail parameter α, use equation C.8 to show[8]

$$\begin{aligned} \tilde{f}_t(\tilde{z}) &= \frac{m_t f_0(\tilde{z}m_t)}{1-F_0(m_t)} && \text{(C.42)} \\ &\propto \frac{m_t L(\tilde{z}m_t)\tilde{z}^{-\alpha-1}m_t^{-\alpha}}{L(m_t)m_t^{-\alpha}} && \text{(C.43)} \\ &\propto \frac{L(\tilde{z}m_t)}{L(m_t)}\tilde{z}^{-\alpha-1} && \text{(C.44)} \end{aligned}$$

[8] Alternatively, define a fat-tailed distribution as: $F_0(z)$ such that $1 - F_0(z) \sim x^{-\alpha}$, where $\alpha > 1$ and $\sim$ denotes asymptotic equivalence. From equation C.10, asymptotically

$$\begin{aligned} \tilde{F}_t(\tilde{z}) &= \frac{1 - \tilde{z}^{-\alpha}m^{-\alpha} - (1 - m^{-\alpha})}{1 - (1 - m^{-\alpha})} && \text{(C.40)} \\ &= 1 - \tilde{z}^{-\alpha} && \text{(C.41)} \end{aligned}$$

Thus, $\tilde{F}_t(\tilde{z})$ is the normalized Pareto distribution as shown in equation C.2,.

If there is perpetual positive growth, i.e., $\lim_{t\to\infty} g_t > 1+\epsilon$ for some $\epsilon > 0$, then by definition $m_\infty \equiv \lim_{t\to\infty} m_t = \infty$. Thus, using the definition of slowly varying,

$$\lim_{t\to\infty} \tilde{f}_t(\tilde{z}) \propto \tilde{z}^{-\alpha-1} \tag{C.45}$$

Therefore, from the stationarity of equation C.21, in any economy with perpetual growth and a power law initial distribution, the asymptotic growth factor is the solution to equation A.21.

Conversely, to derive conditions where growth stops, define $z_{max} \equiv \max \text{support}\,\{F_0\}$. If $z_{max} < \infty$, then growth must stop, as eventually $\lim_{t\to\infty} g_t > 1+\epsilon$ implies $m_\infty = \infty$, contradicting $m_t \leq z_{max}$ for all t, as must be due to the truncation law of motion.

Finally, consider the case where $z_{max} = \infty$ and the initial distribution is not a power law. Note that if there is an equilibrium with no growth at any one point in time, then there is no growth in the limit. At such a point in time, $g = 1$, $r = 1/\beta - 1$, and $\tilde{V}(\tilde{z}) = (1-\tau)\frac{1+r}{r}\tilde{z}$ from equation C.33. Moreover, since no agents choose to search, from equation C.21,

$$(1-\tau)\tilde{z} + \tfrac{1}{1+r} g\tilde{V}(\tilde{z}/g) \geq \varsigma\tilde{z} + \tfrac{1}{1+r} g \tfrac{1}{1-\tilde{F}(g)} \int_g^\infty \tilde{V}(\tilde{z}'/g)\tilde{f}(\tilde{z}')\mathrm{d}\tilde{z}' \tag{C.46}$$

To determine whether equation C.46 can hold with equality, evaluate this inequality at the indifference point, assumed above to be $\tilde{z} = g = 1$

$$1 - \tau - \varsigma \geq \tfrac{1}{1+r}\left(\int_1^\infty \tilde{V}(\tilde{z}')\tilde{f}(\tilde{z}')\mathrm{d}\tilde{z}' - \tilde{V}(1)\right) \tag{C.47}$$

From an initial condition, to find bounds on m_∞, the minimum of support of the asymptotic distribution where firms would choose not to upgrade and growth would stop, substitute equations C.8 and C.33 into C.47 to define

$$\bar{m} \equiv \inf\left\{ m \,|\, \left(\tfrac{1-\tau-\varsigma}{1-\tau}\right)\left(\tfrac{1}{\beta} - 1\right) + 1 \geq \int_1^{z_{\max}/m} \tilde{z}' \frac{m f_0(m\tilde{z}')}{1 - F_0(m)} \mathrm{d}\tilde{z}' \right\} \tag{C.48}$$

From equation C.48, a sufficient condition for the economy to reach an asymptotic maximum size is if a root m exists to the following equation,

$$\left(\tfrac{1-\tau-\varsigma}{1-\tau}\right)\left(\tfrac{1}{\beta} - 1\right) + 1 = \int_1^{z_{\max}/m} \tilde{z}' \frac{m f_0(m\tilde{z}')}{1 - F_0(m)} \mathrm{d}\tilde{z}' \tag{C.49}$$

Since the distribution is not a power law, equation C.49 has a root if $\left(\tfrac{1-\tau-\varsigma}{1-\tau}\right)\left(\tfrac{1}{\beta} - 1\right) + 1 > 1$.[9] Given the assumption in Proposition PT.2, this is true for any $0 < \beta < 1$ and $0 < \tau - \varsigma < 1$. Thus, when $z_{max} = \infty$ and F_0 is such that $\lim_{m\to\infty} \frac{\mathbb{E}[z|z>m]}{m} = 1$, there is a terminal, maximum scale of the economy.

The bound is on the current state m for the firm decisions rather than on the actual terminal value of m_∞. Due to the discreteness of time, in general $m_\infty \neq \bar{m}$. If $m_\infty < \bar{m}$, then the incentives for technology adoption were insufficient for a final step towards $\bar{m}$. It is conceivable that an equilibrium exists where $m_{T-1} < \bar{m}$ with sufficient incentives for adoption that generate growth factor $g_{T-1} > 1$ such that $g_{T-1}m_{T-1} = m_T \geq \bar{m}$. However, from T onward no further growth would occur.

□

[9]Note that if F_0 were a power law, then $\lim_{m\to\infty} \frac{\mathbb{E}[z|z>m]}{m} = \frac{\alpha}{\alpha-1} > 1$, independent of m. Taking the limit of $\beta \to 1$ in equation C.49 gives $1 = \frac{\alpha}{\alpha-1} > 1$. This contradiction shows that there always exist parameters such that power law initial conditions have asymptotic growth.

Appendix D Unconditional Draws

D.1 The Firm Problem

Having upgrading firms draw from the conditional distribution of producing firms simplifies the problem and changes the growth rate quantitatively, but does not qualitatively change the growth mechanism. If upgrading firms received a draw from the unconditional productivity distribution, the ability to meet low productivity agents would lower the equilibrium growth rate by allowing congestion effects as firms may take several draws before they successfully upgrade.

In this section, instead of drawing from the distribution $F_t(z|z > m_{t+1})$, firms draw directly from the unconditional $F_t(z)$ distribution. In that case, the firm may choose to reject a draw if it is lower than its current z. The cost or subsidy of search includes foregone production as well as value proportional to the size of the economy or expected draw. For simplicity, assume that it is proportional to the search threshold in the economy, m_{t+1}.[10] Modifying equation A.1 gives

$$V_t(z) = \max\left\{(1-\tau)z + \tfrac{1}{1+r_t}V_{t+1}(z), \varsigma m_{t+1} + \tfrac{1}{1+r_t}\int_0^\infty V_{t+1}(\max\left\{z', z\right\})dF_t(z')\right\} \tag{D.1}$$

Assume in equilibrium that m_t is increasing (which must be verified). Then firms who search at time t and draw below the current search threshold will search again next period. The probability that the firm draws a z' below the current search threshold is $F_t(m_{t+1})$. Define the gross value of search at time t as

$$W_t \equiv \tfrac{1}{1+r_t}\int_0^\infty V_{t+1}(\max\left\{z', z\right\})\mathrm{d}F_t(z') \tag{D.2}$$

If firms draw below m_{t+1}, they will search next period. Split the integral into the conditional probability distributions above and below m_{t+1} and substitute the net value of search to yield

$$= \tfrac{1}{1+r_t}\left[(1-F_t(m_{t+1}))\int_{m_{t+1}}^\infty V_{t+1}(z')\frac{f_t(z')}{1-F_t(m_{t+1})}\mathrm{d}z' + F_t(m_{t+1})(W_{t+1} + \varsigma m_{t+2})\right] \tag{D.3}$$

To simplify the problem, define $\hat{F}_t(z)$ as the distribution conditional on $z > m_t$.

$$\hat{f}_t(z) \equiv \tfrac{f_t(z)}{1-F_t(m_t)}, \qquad \min \text{support}\left\{\hat{f}_t\right\} = m_t \tag{D.4}$$

$$\hat{F}_t(z) \equiv \tfrac{F_t(z)-F_t(m_t)}{1-F_t(m_t)} \tag{D.5}$$

Recall that the total number of searchers at time t is $S_t = F_t(m_{t+1})$. Define the total number of searchers who were left behind by obtaining "bad" draws as

$$\bar{S}_t \equiv F_t(m_t) \tag{D.6}$$

Note that conditioning the $F_t(z)$ distribution at or above m_t is equal to conditioning the $\hat{F}_t(z)$ distribution at or above m_t, as it is simply two successive truncations. Then, equation D.1 simplifies to

$$\begin{aligned} V_t(z) = \max\Big\{&(1-\tau)z + \tfrac{1}{1+r_t}V_{t+1}(z), \\ &\varsigma m_{t+1} + \tfrac{1}{1+r_t}(1-S_t)\int_{m_{t+1}}^\infty V_{t+1}(z')\frac{\hat{f}_t(z')}{1-\hat{F}_t(m_{t+1})}\mathrm{d}z' + \tfrac{1}{1+r_t}S_t(W_{t+1} + \varsigma m_{t+2})\Big\} \end{aligned} \tag{D.7}$$

[10] An alternative specification is to have the cost/subsidy proportional to the expected draw, conditional on acceptance: $\varsigma\mathbb{E}_t\left[z|z > m_{t+1}\right]$. While a cost proportional to z is possible, it complicates this setup since it requires keeping track of the equilibrium distribution of failed searchers. Economically, as all firms have the same expected value of a draw, independent of their z, it makes sense that the cost/subsidy would also be independent of their z beyond foregone production.

And the gross value of search is

$$W_t = \tfrac{1}{1+r_t}\left[(1-S_t)\int_{m_{t+1}}^{\infty} V_{t+1}(z')\frac{\hat{f}_t(z')}{1-\hat{F}_t(m_{t+1})}\mathrm{d}z' + S_t(W_{t+1}+\varsigma m_{t+2})\right] \tag{D.8}$$

Using the alternative definition of W_t and cost/subsidy ςm_{t+1}, follow the steps in Section B to find the value function on a BGP as

$$V_t(z) = (1-\tau)\tfrac{1+r}{r}\left(1-\left(\tfrac{1}{1+r}\right)^{\xi_t(z)}\right)z + \left(\tfrac{1}{1+r}\right)^{\xi_t(z)}\left(W_{t+\xi_t(z)}+\varsigma m_{t+\xi_t(z)+1}\right) \tag{D.9}$$

$$= (1-\tau)\tfrac{1+r}{r}\left(1-\left(\tfrac{1}{1+r}\right)^{s}\right)z + \left(\tfrac{1}{1+r}\right)^{s} m_0 g^{t+s}\left(W+g\varsigma\right),\ z\in[m_0g^{t+s}, m_0g^{t+s+1}] \tag{D.10}$$

D.2 Law of Motion

The mass of searchers at any point in time, S_t, is now the mass of firms with poor draws in the past plus the new firms that fall below the threshold.

$$S_t = F_t(m_t) + (F_t(m_{t+1}) - F_t(m_t)) \tag{D.11}$$

Using the definitions in equation D.5 and D.6

$$S_t = \bar{S}_t + (1-\bar{S}_t)\hat{F}_t(m_{t+1}) \tag{D.12}$$

The law of motion for $\bar{S}_t$ includes the total number of searchers who draw below m_{t+1} such that

$$\bar{S}_{t+1} = S_tF_t(m_{t+1}) = S_t^2 \tag{D.13}$$

$$\bar{S}_{t+1} = [\bar{S}_t + (1-\bar{S}_t)\hat{F}_t(m_{t+1})]^2 \tag{D.14}$$

Conditional on drawing above m_{t+1}, the draws are in proportion to the distribution truncated at m_{t+1}, as in equation $PT.3$. For this reason, even though the mass of firms below m_{t+1} is not invariant, the truncated distribution is independent of the particular mass in S_t. Hence, the law of motion for the right tail is similar to that in equation $PT.4$,

$$\hat{f}_t(z) = \frac{f_0(z)}{1-F_0(m_t)} \tag{D.15}$$

D.3 Normalization

The normalization follows Section C closely, except that the left truncated distribution $\hat{F}_t(z)$ is normalized instead of the unconditional distribution $F_t(z)$ (i.e., $\hat{F}_t(z) \equiv \tilde{F}_t(\frac{z}{m_t})$). From equations D.12, D.14, C.37, and C.38

$$S_t = \bar{S}_t + (1-\bar{S}_t)\tilde{F}_t(g_t) \tag{D.16}$$

$$\bar{S}_{t+1} = [\bar{S}_t + (1-\bar{S}_t)\tilde{F}_t(g_t)]^2 \tag{D.17}$$

$$Y_t = m_t(1-S_t)\int_{g_t}^{\infty}\tilde{z}\tilde{f}_t(\tilde{z})\mathrm{d}\tilde{z} \tag{D.18}$$

$$\tfrac{1}{1+r_t} = \beta\left(g_t\frac{1-S_{t+1}}{1-S_t}\frac{\int_{g_{t+1}}^{\infty}\tilde{z}\tilde{f}_{t+1}(\tilde{z})\mathrm{d}\tilde{z}}{\int_{g_t}^{\infty}\tilde{z}\tilde{f}_t(\tilde{z})\mathrm{d}\tilde{z}}\right)^{-\gamma} \tag{D.19}$$

Substituting the normalizations into equation D.7,[11]

$$m_t\tilde{V}_t(\tfrac{z}{m_t}) = \max\Big\{(1-\tau)z + \tfrac{1}{1+r_t}m_{t+1}\tilde{V}_{t+1}(\tfrac{z}{m_{t+1}}),$$
$$\varsigma m_{t+1} + (1-S_t)\tfrac{1}{1+r_t}\tfrac{1}{1-\tilde{F}_t\left(\frac{m_{t+1}}{m_t}\right)}\int_{m_{t+1}}^{\infty} m_{t+1}\tilde{V}_{t+1}\left(\tfrac{z'}{m_{t+1}}\right)\tfrac{1}{m_t}\tilde{f}_t\left(\tfrac{z'}{m_t}\right)\mathrm{d}z'\Big\} + S_t\tfrac{1}{1+r_t}m_{t+1}\tilde{V}_{t+1}(1) \tag{D.20}$$

Following the same simplifications as in equation C.19,

$$\tilde{V}_t(\tilde{z}) = \max\Big\{(1-\tau)\tilde{z} + \tfrac{1}{1+r_t}g_t\tilde{V}_{t+1}(\tilde{z}/g_t),$$
$$\varsigma g_t + (1-S_t)\tfrac{1}{1+r_t}g_t\int_{g_t}^{\infty}\tilde{V}_{t+1}(\tilde{z}'/g_t)\tfrac{\tilde{f}_t(\tilde{z}')}{1-\tilde{F}_t(g_t)}\mathrm{d}\tilde{z}' + S_t\tfrac{1}{1+r_t}g_t\tilde{V}_{t+1}(1)\Big\} \tag{D.21}$$

At the indifference point g_t, equate and simplify equation D.21,

$$1-\tau-\varsigma = (1-S_t(g_t))\tfrac{1}{1+r_t(g_t)}\left(\int_{g_t}^{\infty}\tilde{V}_{t+1}(\tilde{z}'/g_t)\tfrac{\tilde{f}_t(\tilde{z}')}{1-\tilde{F}_t(g_t)}\mathrm{d}\tilde{z}' - \tilde{V}_{t+1}(1)\right) \tag{D.22}$$

where $S_t(g_t)$ and $r_t(g_t)$ are defined in equations D.16 and D.19. Comparing equation D.22 to C.20 shows the role of the congestion in changing the incentives for search.

D.4 Stationary Equilibrium with Positive Growth

Using equation D.10 to modify equations C.33 and C.34,

$$\tilde{V}(\tilde{z}) = (1-\tau)\tfrac{1+r}{r}\left(1-\left(\tfrac{1}{1+r}\right)^s\right)\tilde{z} + \left(\tfrac{1}{1+r}\right)^s g^s\left(W+g\varsigma\right),\ \tilde{z}\in[g^s, g^{s+1}] \tag{D.23}$$
$$\tilde{V}(g^s) = (1-\tau)\tfrac{1+r}{r}\left(1-\left(\tfrac{1}{1+r}\right)^s\right)g^s + \left(\tfrac{1}{1+r}\right)^s g^s\left(W+g\varsigma\right),\ \text{for } s\geq 0 \tag{D.24}$$

To solve for the balanced growth path, find an implicit function of g by substituting out W, r, S, and $\bar{S}$

At $\tilde{z}=g$ in equation D.23, equate the $s=0$ and $s=1$ cases using continuity of the value function

$$W + g\varsigma = (1-\tau)g + \tfrac{1}{1+r}g(W+g\varsigma) \tag{D.25}$$

Solving for W,

$$W = \frac{(1-\tau-\varsigma(1-g/(1+r)))g}{1-g/(1+r)} \tag{D.26}$$

From equation D.19, on a BGP

$$r = g^{\gamma}/\beta - 1 \tag{D.27}$$

[11]For the alternative cost,

$$\varsigma\mathbb{E}_t\left[z|z>m_{t+1}\right] = \varsigma\int_{m_{t+1}}^{\infty} z\tfrac{f_t(z)}{1-F_t(m_{t+1})}\mathrm{d}z$$
$$= \varsigma(1-\bar{S}_t)\int_{m_{t+1}} z\hat{f}_t(z)\mathrm{d} = m_t\varsigma(1-\bar{S}_t)\int_{g_t}^{\infty}\tilde{z}\tilde{f}_t(\tilde{z})\mathrm{d}\tilde{z}$$

Therefore, on a balanced growth path the costs are a constant proportion of m_t, and the cost term in equation D.21 for the Pareto distribution and a BGP is $\varsigma(1-S)g^{-\alpha}$.

With the normalized Pareto distribution $\tilde{F}(\tilde{z})$ from equation C.15, equation D.16 and D.17 becomes

$$S = \bar{S} - (1-\bar{S})(1-g^{-\alpha}) \tag{D.28}$$
$$\bar{S} = \left[\bar{S} + (1-\bar{S})(1-g^{-\alpha})\right]^2 \tag{D.29}$$

Solving for $\bar{S}$ and then substituting to solve for S

$$\bar{S} = (g^\alpha - 1)^2 \tag{D.30}$$
$$S = g^\alpha - 1 \tag{D.31}$$

To get the final equation in g, insert the normalized Pareto distribution from equation C.15 into equation D.22, and substitute for $\tilde{V}(1)$ from D.24

$$1-\tau-\varsigma = (1-S)\tfrac{1}{1+r}\left(\alpha g^\alpha \int_g^\infty \tilde{V}(\tilde{z}'/g)\tilde{z}'^{-1-\alpha}\mathrm{d}\tilde{z}' - W - g\varsigma\right) \tag{D.32}$$

Use the piecewise linearity of equation D.23 to turn this integral into an infinite sum

$$\int_g^\infty \tilde{V}(\tilde{z}'/g)\tilde{z}'^{-1-\alpha}\mathrm{d}\tilde{z}' = \sum_{s=1}^\infty \int_{g^s}^{g^{s+1}} \left[(1-\tau)\tfrac{1+r}{r}\left(1-\left(\tfrac{1}{1+r}\right)^s\right)\tilde{z}/g + \left(\tfrac{1}{1+r}\right)^s g^s\left(W+g\varsigma\right)\right]\tilde{z}^{-1-\alpha}\mathrm{d}\tilde{z} \tag{D.33}$$

$$= (1-\tau)\tfrac{1+r}{r}/g\sum_{s=1}^\infty\left(1-\left(\tfrac{1}{1+r}\right)^s\right)\frac{g^{s-(s+1)\alpha}\left(g^\alpha - g\right)}{\alpha-1}$$
$$+ (W+g\varsigma)\sum_{s=1}^\infty\left(\tfrac{1}{1+r}\right)^s g^s\,\frac{g^{-\alpha(s+1)}\left(g^\alpha-1\right)}{\alpha} \tag{D.34}$$

$$= \frac{g^{-\alpha}\left(-g(g\varsigma+W)(\alpha-1) + g^\alpha(g(g\varsigma+W)(\alpha-1)+(1+r)\alpha-(1+r)\alpha\tau)\right)}{\left(-g+g^\alpha(1+r)\right)(\alpha-1)\alpha} \tag{D.35}$$

Substituting this integral, S, W, and r into D.32 and simplifying gives an implicit equation for g

$$\frac{1-\tau-\varsigma}{1-\tau} = \beta g^\alpha\frac{(2-g^\alpha)(\frac{\alpha}{\alpha-1}-g)}{g^{\alpha+\gamma}-\beta g} \tag{D.36}$$

D.5 Stationary Equilibrium with No Growth

For the distributions, including all with finite support, where $\lim_{t\to\infty} g_t = 1$, follow the steps in Section C.5 to find the bound m_∞ at which growth stops,

$$\bar{m} \equiv \inf\left\{m \,|\, \left(\tfrac{1-\tau-\varsigma}{1-\tau}\right)\left(\tfrac{1}{\beta}-1\right)+1 \geq \int_1^{z_{\max}/m}\tilde{z}'\frac{mf_0(m\tilde{z}')}{1-F(m)}\mathrm{d}\tilde{z}'\right\} \tag{D.37}$$

A root may not exist if there is no initial growth at m_0. Otherwise, $\bar{m}$ is a root to the following equation

$$\left(\tfrac{1-\tau-\varsigma}{1-\tau}\right)\left(\tfrac{1}{\beta}-1\right)+1 = \int_1^{z_{\max}/\bar{m}}\tilde{z}'\frac{\bar{m}f_0(\bar{m}\tilde{z}')}{1-F_0(\bar{m})}\mathrm{d}\tilde{z}' \tag{D.38}$$

For non-monotone distributions, it is possible for there to be multiple roots, and m_∞ would be the smallest root.

Appendix E Numerical Algorithm with Unconditional Draws

The following numerically computes a dynamic equilibrium of the economy developed in Section D. The use of unconditional rather than conditional draws adds numerical stability to the algorithm when on or very close to the balanced growth path.

E.1 Setup and Definitions

Setup the following initial and terminal values

- Given an initial condition for $f(z)$:
  1. Choose an $m_0 > 0$. This is arbitrarily chosen as long as in equilibrium $m_1 > m_0$. If $m_1 \leq m_0$ after the calculations, then lower m_0.
  2. Initialize the number left beyond to be $\bar{S}_0 = F(m_0)$
  3. Get the normalized version of the truncated pdf from equation C.8: $\tilde{f}(\tilde{z}) = \frac{m_0 f(\tilde{z} m_0)}{1-F(m_0)}$ by equation C.8.
- Choose a large terminal time T.
- If the system is converging towards a balanced growth path, then the asymptotic value function is $\tilde{V}_T(\tilde{z})$ from equation D.23. If the system is converging towards $g = 1$, then the terminal value will simply be the normalized value of production in perpetuity, $\tilde{V}_T(\tilde{z}) = (1-\tau)\frac{1+r}{r}\tilde{z}$.
  - The closed-form value function in equation D.23 can be compared against naive value function iteration of the stationary version of equation D.21. For the fixed point, as the value function is known to be piecewise linear, cubic-splines and other smooth interpolation methods should not be used. Value function iteration is slow but safe.
  - Due to the accumulation of numerical errors from integration, these may be different at 3-5 significant digits. For larger growth rates, this small difference can compound geometrically, and change dynamics close to the terminal T. When calculating dynamics in these cases, it often makes sense to use the solution for $\tilde{V}_T(\tilde{z})$ from value function iteration as it is more consistent with the backwards induction used in the rest of the algorithm.
- Choose a number M of future indifference points for approximating the value function. At any given t, after M points the value function is assumed to be linear, as shown in equation B.6.
- Choose an initial guess for the sequence of growth factors $\vec{g} \equiv \{g_t\}_{t=0}^{T-1+M}$ where $g_t = g_\infty$ for $T \leq t \leq T-1+M$. This pads the guess of growth factors with the asymptotic growth factor to ensure that there are always M future points in the approximation of the value function (i.e. $M+1$ total points with $\tilde{z} = 1$).

At a particular time, for a particular $\vec{g}$, use equation C.25 with $\vec{g}$ to calculate the set of points to use in the approximation of the value function

$$\vec{m}_t = \left\{1, \prod_{t'=0}^{0} g_{t+t'}, \prod_{t'=0}^{1} g_{t+t'}, \ldots, \prod_{t'=0}^{M-1} g_{t+t'}\right\} \tag{E.1}$$

For example if $M = 3$, $\vec{m}_5 = 1, g_5, g_5 g_6, g_5 g_6 g_7$.

E.2 Iterative Algorithm

Given a current guess $\vec{g}$

1. Calculate the sequence of normalized densities $\tilde{f}_t(\tilde{z})$ for $t = 0, \ldots, T$.
   (a) For the baseline model, first calculate the set of (unnormalized) indifference points using equation C.24.

$$\{m_t\}_{t=0}^{T} = \left\{m_0 \prod_{t'=0}^{t-1} g_{t'}\right\}_{t=0}^{T} \tag{E.2}$$

(b) Use equation C.8 to calculate the normalized sequence of distributions, $\left\{\tilde{f}_t(z)\right\}$.[12] For numerical stability in the right tail, it may make sense to use the expression $A/B = e^{\log(A)-\log(B)}$

$$\tilde{f}_t(\tilde{z}) = \exp\left(\log(m_t) + \log(f(\tilde{z}m_t)) - \log(1 - F(m_t))\right) \tag{E.3}$$

$$\tilde{F}_t(\tilde{z}) = \exp\left(\log(F(m_t\tilde{z}) - F(m_t)) - \log(1 - F(m_t))\right) \tag{E.4}$$

(c) Alternatively, for the particular $F_0(z)$, the functions for the truncation at m, $\tilde{f}_m(\tilde{z}; m)$ and $\tilde{F}_m(\tilde{z}; m)$ could be given directly in accordance with equation C.8 and C.10. This method should be used if the expression simplifies to remove m_t from the denominator. Otherwise, for many distributions and large m, $1 - F(m)$ and $f(\tilde{z}m)$ can both reach 0 within the computers level of precision, yielding imprecise or 0/0 values for $\tilde{f}_t(\tilde{z})$ at large t.

(d) From $\bar{S}_0$, use equations D.17 and D.16 to calculate the mass of searchers and those left behind, $\left\{\bar{S}_t, S_t\right\}$.

2. Calculate the value function backwards for $t = T - 1, \ldots 0$ in order to solve for the reservation productivities, $\{g'_t\}$. Start with the analytical value $\tilde{V}_T(\tilde{z})$ and use the calculated $\left\{\tilde{f}_t, r_t, S_t\right\}$.

   (a) Calculate the sequence of points to evaluate as $\vec{m}_t$ using equation E.1.

   (b) Calculate the value function for each value in $\vec{m}_t$ using equation D.21.

   - This requires numerical integration over the distribution $\tilde{f}_t(\tilde{z})$ using the previously calculated $\tilde{V}_{t+1}$.[13]
   - Set $\tilde{V}_t(\tilde{z})$ as the piecewise linear function between the points $\vec{m}_t$ with the calculated values for use in time $t - 1$ calculation. The value function should be extrapolated linearly beyond the M points.

   (c) Calculate the indifference point with this value function to find g'_t using equation D.22.

   - For numerical stability when converging towards the BGP, it is crucial when calculating the root that $S_t(g)$ and $r_t(g)$ also move to solve the indifference point.
   - To implement this, for the given $\bar{S}_t$, find a root to equation D.22 using equations D.16 and D.19. In the normalized space, this root is the new g_t.
   - The value function is piecewise linear, so splines and many other approximations are inappropriate. While the kinks should be at $\vec{m}_t$, for numerical stability in the backwards induction it is helpful to add a finer grid between points.

3. If $\{g'_t\}_{t=0}^{T-1+M}$ is close in norm to $\vec{g}$, stop iterating.[14]

   - Otherwise, using the sequence $\{g'_t\}_{t=0}^{T-1+M}$, update the guess $\vec{g}$. For example, use a linear combination of the two for the new $\vec{g}'$.

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[12] For different models that imply different laws of motion, replace C.8 with the new law of motion, and use $\{m_t\}_{t=0}^{T}$ to sequentially calculate an approximation of $\left\{\tilde{f}_t(z)\right\}$.

[13] In order to calculate this numerically, a finer grid than $\vec{m}_t$ will be required. Since $\tilde{V}_{t+1}$ is piecewise linear and kinked, Simpsons rule and similar quadrature approaches are not appropriate unless calculated piecewise between each interval in $\vec{m}_{t+1}$. Note $\lim_{\tilde{z}\to\infty} \tilde{V}'_{t+1}(\tilde{z}) = (1 - \tau)\frac{1+r}{r}$ and $\lim_{\tilde{z}\to\infty} \tilde{f}_t(\tilde{z}) = 0$. Alternatively, Gauss-Laguerre quadrature can be used for the infinite right tail integral, or adaptive quadrature routines such as Matlab's *integral* may be used.

[14] At termination, the $\tilde{f}_T(\tilde{z})$ should be checked to be close to $\tilde{f}_\infty(\tilde{z})$ as calculated analytically from equation C.12.

E.3 Baseline Calibration

In the baseline model in Proposition $PT.1$, the only parameters are α, β, and γ. The cost of search is in foregone production, so in order to calibrate to a particular growth rate, the length of a time period must be calibrated to change the search costs. To calibrate to reasonable growth rates, this may lead to long period lengths, and consequently low β if α is low. In order to establish a yearly period length, negative values of ς in equation D.1 provide multiples of the time periods for lost production. With this variation of the model, the calibrated values are:

- $\gamma = 1$, $\beta = 0.95$: Targets annualized interest rates
- $\alpha = 1.5$: A compromise between estimates of the far right tail of the firm productivity distribution, which is often estimated to be low (e.g., 1.1 if ignoring the lower tail) and the need to fit the Pareto to the whole of the distribution of operating firms
- $\tau = 0.3$: 30% tax rate on earnings
- $\varsigma = -12$: Note that the mean of the normalized Pareto with $\alpha = 1.5$ is 3. With yearly time periods and the additional cost of lost production for the period, it takes approximately 5 years of profits for an average producer to break even when upgrading their technology. See Perla, Tonetti, and Waugh (2013) for a version of this cost function paid in labor and final goods at equilibrium prices.

With the above calibration, the asymptotic annualized growth rate is 3.28%.

E.4 Fréchet Example

For a Fréchet initial condition with pdf and cdf

$$f(z) = e^{-\left(\frac{z}{s}\right)^{-\alpha}} \alpha s^{\alpha} z^{-1-\alpha} \tag{E.5}$$

$$F(z) = e^{-\left(\frac{z}{s}\right)^{-\alpha}} \tag{E.6}$$

Using equation D.15, the normalized pdf is

$$\tilde{f}_t(\tilde{z}) = \frac{e^{-\left(\frac{m}{s}\tilde{z}\right)^{-\alpha}} \left(\frac{m}{s}\right)^{-\alpha}}{1 - e^{-\left(\frac{m}{s}\right)^{-\alpha}}} \alpha \tilde{z}^{-\alpha-1} \tag{E.7}$$

From Section C.5, the economy will converge to the balanced growth path with the shape parameter α, independent of s. Hence, the analytical $\tilde{V}_T(\tilde{z})$ is given in equation C.33. Given the calibrations in Section E.3, with $s = 1$ and $\alpha = 1.5$, the growth dynamics for a Fréchet initial condition is given in Figure 1.

To see the connection between the “thickness” of the distribution and the growth rates, Figure 1 also plots the normalized right tails of the productivity pdfs, $\tilde{f}_t(\tilde{z})$ for $t = 2, 5, 40$, and the asymptotic limit.[15] Due to the equivalence of solving a normalized version of the problem from the normalized distributions, as discussed in Section C, these distributions are sufficient to interpret the search decision of agents and solve for growth rates of the economy. Note that as t increases, the normalized distribution becomes thinner with a smaller expectation and more mass closer to the minimum of support. This decrease in the expected normalized draw creates decreasing growth rates as the economy converges towards the power law tail. Note that by year 40, the normalized distribution conditional on production is nearly Pareto.

[15] This diagram, and a discussion of the economics surrounding it, are also in Figure $PT.2$. While it is not shown here, for other parameter choices, $\tilde{f}_t(\tilde{z})$ could exhibit the increasing left tail of the initial Fréchet distribution for small t.

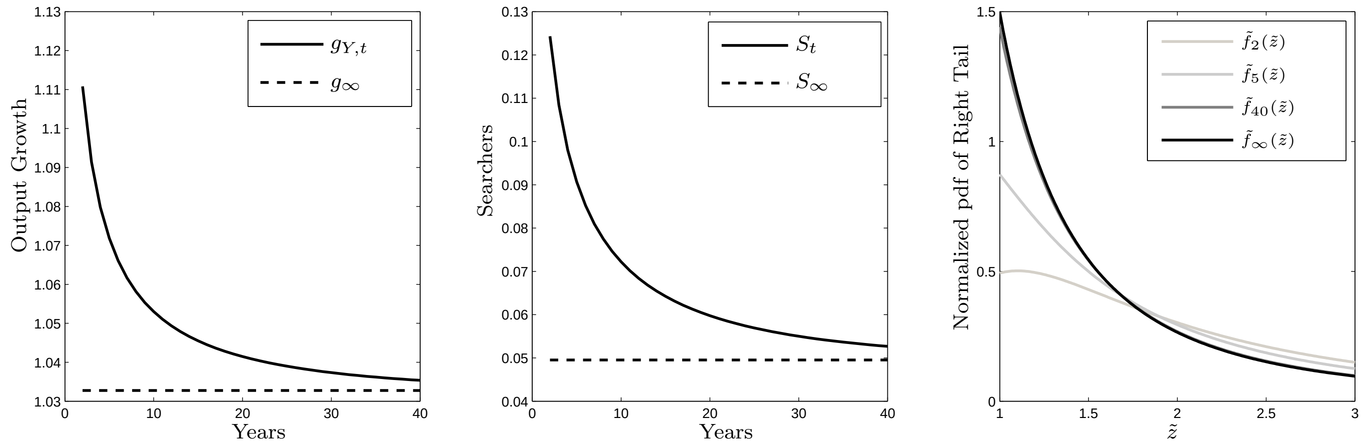

Figure 1: Growth and Normalized Distributions from Fréchet Initial Condition

E.5 Bounded Pareto Example

Since in this model no new technology is ever created through an external R&D process, if the initial distribution has finite support then eventually growth will stop below the frontier, as discussed in Section D.5. However, at any point in time a far off frontier has little effect on the decisions of agents to upgrade their technology. To see this, take the parameters used in Section E.3, but right truncate the Pareto distribution. To illustrate a far off technology frontier, truncate the distribution such that $\max\{\tilde{z}\} = 500$, which ends up decreasing production (and hence the expected value of a draw) by about 4.5% at year 0. Figure 2 shows the results for 100 years of transition dynamics, and compares to the balanced growth path solution with the a Pareto initial condition with the same α (denoted g_u and S_u).

With this, the pdf and cdf with a minimum of support m and maximum of support $\bar{z}$ are

$$f(z) = \frac{\alpha m^\alpha z^{-1-\alpha}}{1 - \left(\frac{m}{\bar{z}}\right)^\alpha} \tag{E.8}$$

$$F(z) = \frac{1 - \left(\frac{m}{z}\right)^\alpha}{1 - \left(\frac{m}{\bar{z}}\right)^\alpha} \tag{E.9}$$

The growth rates start lower than the unbounded case, due almost entirely to the lower expected value in the distribution rather than the bounded support.[16] After about 100 years the growth rate has decreased by less than 1%, reflecting the approaching boundary. However, with the calibrated growth rates of $2 - 3\%$, the technology frontier approaches very slowly and the normalized distribution stays nearly Pareto. Figure 2 shows the productivity distribution after 2 and 400 years. While the normalized maximum of $\tilde{z}$, as calculated by equation C.1, is 500 at $t = 1$ and 6.9197 at $t = 400$, the probability of a large draw changes less drastically. For example, $1 - \tilde{F}_2(3) = 0.1924$ and $1 - \tilde{F}_{400}(3) = 0.1455$.

[16]To see why this drop is significant, note that if $\alpha = 1.5$, the ratio of the expected value of the Pareto distribution bounded below $\tilde{z} = 500$ to the unbounded case is .955. When the tail parameter α is higher, this ratio increases towards unity and the initial growth rates are much closer to the unbounded case. For example, with $\alpha = 2.1$, the ratio of expectations for the case bounded below $\tilde{z} = 500$ is 0.998.

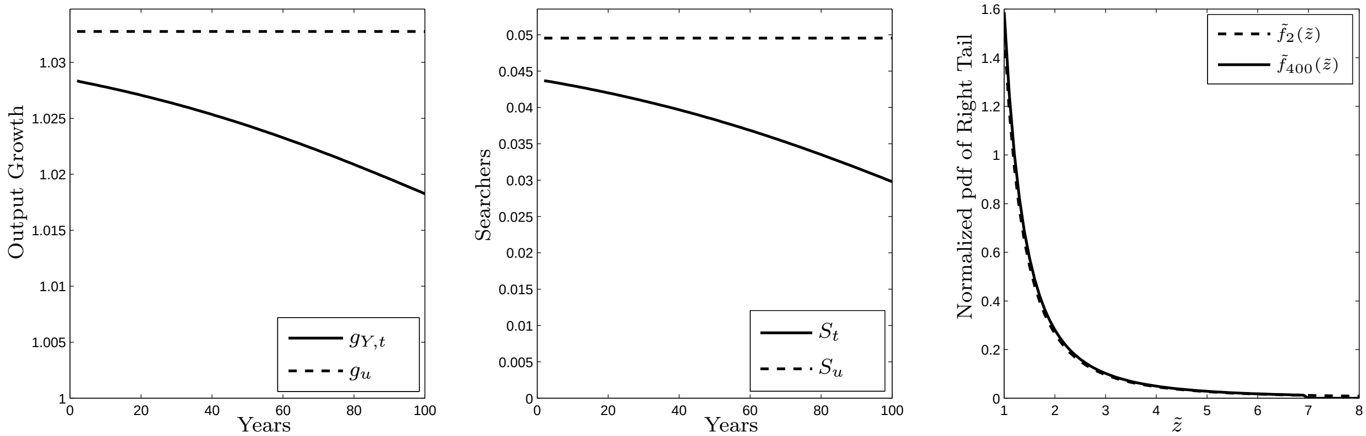

Figure 2: Growth and Normalized Distributions from Bounded Pareto Initial Condition

References

Perla, J., C. Tonetti, and M. E. Waugh (2013): "Equilibrium Technology Diffusion, Trade, and Growth," *NYU mimeo*.