A Model of Product Awareness and Industry Life Cycles*

Laurent Cavenaile† Murat Alp Celik‡ Jesse Perla§ Pau Roldan-Blanco¶

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Abstract

Rapid technological advances in advertising have enabled firms to better target those consumers most likely to buy their products. While more efficient than traditional methods, targeted advertising may significantly limit product market competition. We propose a model of demand as a network, where heterogenous consumers dynamically become "aware" of differentiated products, expanding their choice sets and improving on their possible matches thanks to advertising. As networks become denser, customer misallocation decreases due to better sorting. However, though more intensive targeting can efficiently sort with fewer network connections, it also increases market power by segmenting consumers. We calibrate the model to the United States over a period of time which saw a rapid rise in digital advertising. We find that this rise led to substantially better consumer-firm matches. However, if the targeting technology had not improved during this period, markups would have been lower and welfare higher despite worse sorting.

Keywords: Industry Life Cycle, Advertising, Customer Capital, Information Frictions, Targeting, Choice Sets.

JEL Classification: D40, E20, L10, M30.

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†University of Toronto Scarborough and Rotman School of Management; 105 St. George Street, Toronto, Ontario, M5S 3E6. Contact: laurent.cavenaile@utoronto.ca

‡University of Toronto; 229 Max Gluskin House, 150 St. George Street, Toronto, Ontario, M5S 3G7. Contact: murat.celik@utoronto.ca

§Vancouver School of Economics, University of British Columbia; 6000 Iona Dr, Vancouver, BC V6T 1L4, Canada. Contact: jesse.perla@ubc.ca

¶Banco de España and CEMFI; Calle Alcalá 48, 28014, Madrid. Contact: pau.roldan@bde.es

1 Introduction

Advertising helps firms build a customer base by spreading product awareness. Firms have traditionally employed a mixture of broad-based advertising (e.g., door-to-door sales, billboards along highways, print and radio advertising) and targeted advertising (e.g., buying mailing lists, sending customer catalogues) to accomplish this. While these practices have been used by businesses for over a century, recent radical technological advances and, in particular, the rapid rise of digital advertising over the past two decades, have improved the efficiency of both broad-based and targeted forms of advertising.

The ascent of digital technologies is evident in the aggregate data on media spending. In the United States, online advertising revenue —including search, social media, and mobile— grew from 8.09 billion U.S. dollars in 2000 to 189.3 billion U.S. dollars in 2021.[1] In fact, digital advertising has vastly outgrown any other advertising method. In North America as a whole, the share of total advertising spending accounted for by digital advertising grew from around 4 percent in 2000 to 17 percent in 2010, accelerated to over 57 percent by 2020, and is expected to be 70 percent in 2023.[2]

Initially, digital advertising seemed to largely substitute for existing broad-based technologies, and it only slowly shifted toward more tailored results not previously possible (e.g., it replaced television ads with indiscriminate banner ads, but also let firms pay to be at the top of search results for a particular keyword). But with the rapid expansion of social networking platforms and linked browsing data, digital advertising could suddenly and very efficiently target consumers using the shopping and viewing habits, demographics, and current location of both the individual and their entire social network.[3] All in all, these technological changes in advertising are fundamentally affecting how customers are reached and markets are structured, and are likely to have aggregate welfare consequences.

To study this question, in this paper we develop an information-based general-equilibrium theory of industry dynamics, which we use to explore the macroeconomic implications of improvements in advertising technologies for industry dynamics, competition, and welfare. In the model, a set of consumers seeks to purchase products from a continuum of industries. In each industry there is a finite number of firms producing horizontally differentiated products who compete in prices à la Bertrand. Consumers are heterogeneous in their idiosyncratic preferences for the products

[1] Information from Statista, using data from PwC and IAB: https://www.statista.com/statistics/183816/us-online-advertising-revenue-since-2000/. As of 2022, the United States is the largest digital advertising market globally, accounting for more than one-third of the world's digital advertising revenue. It also has the largest expected growth in revenue: total digital advertising revenue is expected to increase at a compound annual growth rate of 11 percent until 2027 (https://www.statista.com/study/42540/digital-advertising-report/).

[2] Information from Statista, using data from Zenith's Advertising Expenditure Forecasts (December 2021 report): https://www.statista.com/statistics/429036/advertising-expenditure-in-north-america/. The advent of digital advertising is not limited to the United States. In 2022, digital advertising accounted for 67 percent of total advertising revenue worldwide, and is expected to surpass 70 percent by 2025 (information from Statista, using data from GroupM: https://www.statista.com/statistics/375008/share-digital-ad-spend-worldwide/).

[3] Indeed, in the United States in 2021, 63 percent of all digital advertising revenue came from mobile, which makes it possible to accurately track a customer's location. This number is expected to have increased to 72 percent by 2027 (https://www.statista.com/study/42540/digital-advertising-report/).

of the different firms within the industry. To this setup, we introduce information frictions in the form of limited product awareness: at each point in time, consumers are only aware of a subset of all the products that are being produced in each industry.[4] The awareness sets evolve over time endogenously as new connections are formed given advertising strategies. Thus, the model re-imagines demand functions as an endogenous network between consumers and producers, and nests the standard CES demand structure as a limiting case when the network is dense (i.e., when every consumer is aware of every firm).

In equilibrium and in each industry, consumers purchase the product which better matches their preferences given their limited choice sets, taking into account the product's price relative to that of the other products that the consumer is aware of. A firm's demand is thus not only a function of prices along the intensive margin (i.e., a firm's incumbent customer demands less of the product if the price is higher), but also along the extensive margin (a price change may induce those customers with other options to switch all of their demand to competitors). We label this latter component of demand as *sorting*. In equilibrium, pricing strategies respond to the firms' current network of customers, and to that of their competitors within the industry. Properties of the network then determine each product's potential customer base, the efficiency of sorting consumers to their preferred purchases, and the degree of market power inherent in an industry.

Within this framework, we model advertising as a technology that affects the connections that are formed between consumers and firms over the industry's life cycle. There are two dimensions to advertising in the model. The first one directly affects the rate at which new network connections are formed, e.g., the rate at which consumers can contact firms that were not previously in their awareness set. Advertising therefore affects the speed at which awareness sets expand, thereby allowing consumers to sort into more preferred products over time and reducing consumer misallocation. The second dimension of advertising relates to the degree to which advertising is targeted. Targeting allows firms to distort the consumer type that they meet, allowing them to reach consumers with higher idiosyncratic preference for their product earlier on. We label this second dimension of advertising as *targeting*, as it shows up as a shifter in firm-level demand along the intensive margin: when the product is a better fit for the incumbent customer's preference, demand is higher. In sum, advertising fulfills two roles in the model: it raises product awareness, and it permits better matches with fewer connections.

These two different dimensions of advertising inexorably interact with the firm's pricing decisions. On the one hand, a higher contact rate allows for faster network formation and, therefore, yields lower consumer misallocation through better sorting: as consumers have more alternatives to choose from, they gain faster access to products that they may prefer over those they are currently purchasing. As networks expand, competition intensifies and markups decline.[5] On the other hand,

[4]We label information frictions for consumers within the network with the term "limited awareness," following the empirical literature on this topic (e.g., Goeree (2008) and De Los Santos, Hortaçsu and Wildenbeest (2012)). Limited awareness captures frictions such as a consumer having no knowledge of a product's existence, the idiosyncratic match to their preferences, and/or the location of a distributor.

[5]More precisely, we show that firm markups are inversely related to the elasticity of the sorting (or extensive-margin)

a high degree of targeting allows firms to find consumers who are less likely to switch to new products as the network becomes denser. This results in the firm extracting a higher match surplus while being able to maintain high prices. As targeting allows firms to decrease the likelihood of forming network connections which would not result in a sale, this advertising technology lets firms segment their market and more easily find the most profitable and least price-sensitive consumers.

In spite of this rich micro-level heterogeneity and various competing forces, we show that the model can be aggregated into a representative-agent neoclassical growth economy. This result is particularly useful because it allows us to quantify welfare losses from information frictions by means of a sufficient statistic, a wedge to aggregate TFP, which encodes (i) the degree of markup dispersion within the industry; (ii) the degree of connectedness of awareness sets; (iii) match quality, a term which encompasses both the degree of sorting of consumers to preferred products and the amount of targeting; and (iv) general-equilibrium effects determining the total measure of product categories available in the economy.

Using this representation, we calibrate the parameters of our model with the goal of quantifying the macroeconomic implications of the rise in digital advertising for consumer sorting, competition, and welfare. At first glance, digital advertising technology and the related shift to intensely targeted advertising appear unambiguously beneficial. Firms prefer to efficiently market to those consumers most likely to purchase their product, and consumers prefer both more choice and to be contacted about those products they are likely to purchase.[6] Closer inspection, however, reveals that intensely targeted advertising may fail to decrease market power through the industry life cycle and lead to less product categories in equilibrium, mitigating some of those benefits.

To understand these tensions, the stationary solution of the model is calibrated separately to two time periods, 2005 and 2014, which saw a rapid increase in the share of digital advertising in total advertising spending. We interpret the advent of digital advertising in recent years as a rise in the effectiveness of targeting, using the click-through rate of targeted vis-a-vis untargeted digital advertising in the data as a proxy for the return to targeting in the model. Under this interpretation, our calibrations deliver that the rise of digital advertising yielded a decrease in the cost of targeting, but also, though to a lesser extent, in the cost of contacting new consumers. As the cost of targeting became relatively cheaper than that of contacting over the 2005-2014 period, the rise of digital advertising led to a lower contact rate —awareness sets expanded more slowly— but also to a higher degree of targeting —those matches that did form were of higher quality on average, i.e., they correlated more strongly with the consumers' idiosyncratic preferences. In terms of welfare, we find that aggregate TFP increases despite the decrease in the contact rate, due primarily to the strong effects of increased match quality (which combines the effects of targeting and sorting) on

component of demand. In the initial stages of the industry, awareness sets are small and firms exert high market power. But as the industry matures and the consumer becomes aware of more alternatives, extensive-margin demand becomes more elastic since firms face more de-facto competitors, putting downward pressure on markups.

[6] See, e.g., https://www.globalwitness.org/en/blog/do-people-really-want-personalised-ads-online/. As another example, in a study conducted by Adlucent LLC, over 70 percent of responders said they would prefer ads tailored to their interests and shopping habits, see https://www.adlucent.com/resources/blog/71-of-consumers-prefer-personalized-ads/.

overall product demand, raising both consumption and total output, decreasing the profit share, and increasing the labor share.

Finally, we conduct a series of counterfactual exercises to understand what would these effects have been if there had not been an improvement in the advertising technology. Starting from the late calibration, we re-compute the economy's stationary equilibrium assuming that the cost parameters related to advertising are set back to their levels in the early period, but all other parameters remain fixed at their calibrated values for the late period. In this exercise we find that, had there been no changes in advertising costs from 2005 to 2014, firms would have substituted more frequent contacts for a lower level of targeting, leading to a more competitive environment than what the baseline calibration for the late period predicts. While total match quality would have been lower, the increased competition and the positive general equilibrium effect on the measure of product categories would dominate to deliver higher welfare in this counterfactual economy. Thus, consumers would be better off if the advertising technology, and targeting in particular, had not become more efficient over time.

Literature review Our paper relates to several strands of the literature. Most directly, we contribute to a literature in macroeconomics and international trade that studies the implications of customer capital for firm and industry dynamics. Contributions to this literature include Fishman and Rob (2003), Luttmer (2006), Arkolakis (2010, 2016), Dinlersoz and Yorukoglu (2012), Drozd and Nosal (2012), and Gourio and Rudanko (2014a,b).[7] In these models, firms grow via the accumulation of idiosyncratic demand, which the empirical literature has found to be an important determinant of both the overall dispersion of firm sales and the growth dynamics of firms (see e.g., Foster, Haltiwanger and Syverson (2008) for evidence in the manufacturing sector, and Hottman, Redding and Weinstein (2016) and Einav, Klenow, Levin and Murciano-Goroff (2022) for retail markets). Our paper contributes to this literature by showing that it is not just accumulated customers that matter, but also the features of the interconnected network that aggregate to form the firm's customer capital.

Our interpretation for the slow-moving process of demand accumulation is related to the idea that consumers accumulate information slowly about the producers that they can purchase from. Goeree (2008) uses a similar notion of awareness as limited information sets, but in a largely static fashion. Like us, Guthmann (2020a,b) explores the dynamic implications of limited awareness, but through word-of-mouth dynamics among buyers and price-posting strategies on the side of sellers similar to those in Butters (1977). Our main contribution in the limited awareness literature is to provide a connection with advertising choices along two margins, the speed at which new customers are contacted, and the quality of the firm-customer matches via targeting. Moreover, unlike previous papers, we quantitatively analyze how this process shapes industry dynamics and aggregate welfare through its effects on competition and sorting in a general-equilibrium setting.

[7] Some earlier models of advertising are due to Dorfman and Steiner (1954), Butters (1977), Stegeman (1991) and Becker and Murphy (1993). For a survey of the advertising literature in economics, see Bagwell (2007).

Various studies lend support to our assumptions regarding demand formation. First, for there to be significant quantitative consequences of our information friction, networks must remain relatively sparse and choice sets cannot be large. Empirical studies able to connect individuals to choice sets consistently show consumers choose between few options, a surprising finding in light of the rapid advances in advertising technology. For example, De Los Santos *et al.* (2012) find that 35 percent of consumers only visit a single online bookstore during 18 months of data, while Honka and Chintagunta (2016) document average choice sets of size two to three in the market for auto insurance.[8] We rely on these types of findings to model demand as a network in which connections between firms and consumers are formed slowly in response to the advertising decisions, and awareness sets remain relatively small. Moreover, numerous studies have documented an important role empirically for the entry and exit of products into household consumption baskets, e.g., Broda and Weinstein (2010), Argente, Fitzgerald, Moreira and Priolo (2021), and Michelacci, Paciello and Pozzi (2021). Our model can be seen as offering a micro-foundation for these dynamics along the extensive margin of demand.[9] Finally, in our model the prospect of accumulating demand also shapes the incentives to create new products through innovation, as in Cavenaile and Roldan-Blanco (2021) and Ignaszak and Sedláček (2022). Therefore, changes in the advertising technology also affect the number of products that enter the economy, which has consequences for welfare.

Part of the literature has emphasized the role of price dynamics in models with consumer markets, such as in Klemperer (1995), Bergemann and Välimäki (2006), Shi (2016), Roldan-Blanco and Gilbukh (2021), and Rudanko (2022), among others. In our paper, firm prices respond to the firm's desire to form better matches while retaining their existing customers. In particular, in our model incomplete information on the side of buyers has implications for competition and markups, yielding welfare losses due to misallocation in the aggregate. In this sense, we also relate to papers where advertising and customer markets give rise to misallocation through market power. Along these lines, Cavenaile, Celik, Roldan-Blanco and Tian (2022) show that advertising can have beneficial effects on allocative efficiency, thereby alleviating static welfare losses from input misallocation, albeit at the cost of crowding out R&D resources. Relatedly, De Ridder (2022) argues that a more intensive use of intangible investments might give rise to increases in concentration. Afrouzi, Drenik and Kim (2020) show that firms increase market shares through the number of customers but exert market power through non-pricing activities, consistent with our setting where customer accumulation is driven by advertising. We contribute to this literature by proposing a new mechanism for market power coming from the endogenous formation of consumer-firm networks, and from firm-level investments into creating better matches through targeting.

[8] Looking at direct relationships of buyers and sellers using Colombian export data, Eaton, Eslava, Krizan, Kugler and Tybout (2014) also find very small networks, of around 1.5 buyers per exporter and 4 sellers per buyer, on average. Bernard, Dhyne, Magerman, Manova and Moxnes (2022), Fitzgerald, Haller and Yedid-Levi (2022) and Lenoir, Martin and Mejean (2022) also document that the customer margin plays an important role in export markets among Belgian, Irish and French firms, respectively.

[9] Like in our model, Paciello, Pozzi and Trachter (2019) build a theory in which firms respond to changes in their extensive-margin demand, though in that model this occurs because customers can choose to search for other suppliers when faced with a price change.

Finally, we also relate to a small but nascent literature that studies the effects of the rise of digital advertising on the aggregate economy. Similar to our paper, though using a very different setting, Baslandze, Greenwood, Marto and Moreira (2022) focus on how the advent of digital advertising may have welfare implications through an increase in product varieties; Rachel (2022) argues that the emergence of leisure-enhancing technologies, e.g., through media platforms financed by advertising, may have had an adverse effect on hours worked and aggregate TFP; and Greenwood, Ma and Yorukoglu (2021) argue instead that the rise in digital advertising may have had positive effects on welfare because it alleviates an under-provision inefficiency problem in media goods, which are valued by consumers because they increase utility through non-market activities. Complementing these studies, we find that the increase in the returns to targeting has a positive welfare effect via an increase in overall match quality (i.e., consumers having earlier access to products that they value more). However, in counterfactual experiments, we also find that this change in the advertising technology was associated with an increase in the average level of markups and a decrease in new industry creation, offsetting the benefits on welfare.

The remainder of the paper is organized as follows. Section 2 introduces our model of demand as a network, solves for its equilibrium conditions, and shows that the economy aggregates to a neoclassical model in which micro-level information frictions give rise to a wedge on aggregate TFP. Section 3 discusses the estimation of the model and analyzes the implications of the mechanism for industry dynamics. Section 4 discusses our counterfactual experiments, where we analyze the role of changes in the advertising technology for industry dynamics, markups, match quality, and welfare. Section 5 concludes.

2 Model

2.1 Environment

Demographics Time is continuous, runs forever, and is indexed by t. The economy is populated by a measure-one continuum of infinitely-lived and heterogeneous consumers indexed by $j \in [0,1]$, with preferences over a continuum of industries (sometimes called product categories). The measure of industries is endogenous and denoted by $\boldsymbol{M}_t > 0$.[10] Each industry $m \in [0, \boldsymbol{M}_t]$ is populated by the same exogenous number $N \in \mathbb{Z}_+$ of identical single-product firms indexed by $i \in \mathcal{I} \equiv \{1, \dots, N\}$. These firms interact strategically within their industry.

There is a single final good, which is used for consumption and investment. There are three types of investment: in physical capital, in advertising, and in industry creation. Consumers supply labor inelastically in a frictionless labor market at a wage w_t, and they own the stock of physical capital in the economy, which is rented to firms at the perfectly competitive rate R_t^K. Consumers also receive dividends from the firms' profits, and trade in financial assets that pay the interest rate r_t.

[10] Throughout, we use the following notational convention: lowercase letters and symbols refer to firm-level variables, uppercase letters and symbols are for industry-level variables, and uppercase bold ones refer to aggregate variables.

Consumer preferences Consumer $j \in [0, 1]$ maximizes lifetime utility:

$$\int_0^{+\infty} e^{-\rho t} \frac{C_{jt}^{1-\gamma}}{1-\gamma} \mathrm{d}t, \tag{1}$$

where $\rho > 0$ is the time discount rate, $\gamma > 1$ is the coefficient of relative risk aversion, and C_{jt} is individual j's level of consumption. Each individual j can purchase the output produced by the different firms $i \in \mathcal{I}$ within each industry $m \in [0, \boldsymbol{M}_t]$, provided they are aware of the product (as described below). The pair (i, m) uniquely identifies a product.

Consumers are heterogeneous along two dimensions. First, we assume permanent heterogeneous preferences across products, captured by a time-invariant preference shifter $\xi_{imj} > 0$. We make the following assumption regarding the distribution of idiosyncratic preferences:

Assumption 1 (Idiosyncratic preferences)**.** *Preference shifters (ξ_{imj}) for the population of consumers are independent and identically distributed across consumers and industries according to a type-I generalized extreme value (or Gumbel) distribution, with location parameter equal to zero and scale parameter equal to one.*

Assumption 1 ensures, in particular, that the distribution of preferences is independent across industries for the same individual consumer, i.e., $\xi_{imj} \perp \xi_{im'j}, \forall m \neq m'$.

The second dimension over which consumers are heterogeneous is the set of firms that they are aware of within any given industry at each point in time. Consumer j may only purchase goods from the subset $A_{mjt} \subseteq \mathcal{I}$ of firms i in industry m that she is aware of at time t. The evolution of the awareness sets A_{mjt} over time is endogenous (affected by advertising choices), stochastic, and idiosyncratic to each consumer-industry pair.

With these assumptions in place, we define the individual-specific consumption bundle C_{jt} from equation (1) as a CES composite of the consumption of the different products that compose the consumer's awareness set in that industry:

$$C_{jt} = \left[\int_0^{\boldsymbol{M}_t} \left(\sum_{i \in A_{mjt}} \overline{\Gamma} e^{\sigma \xi_{imj}} c_{imjt} \right)^{\frac{\kappa-1}{\kappa}} \mathrm{d}m \right]^{\frac{\kappa}{\kappa-1}}, \tag{2}$$

where $\kappa > 1$ is the elasticity of substitution between different industries, $\sigma \in \left(0, \frac{1}{\kappa-1}\right)$ measures the degree of preference differentiation between firms within an industry, and $\overline{\Gamma} \equiv \Gamma(1 - \sigma(\kappa - 1))^{\frac{1}{1-\kappa}}$ is a normalizing constant, where $\Gamma(\cdot)$ is the Gamma function.[11] Equation (2) shows that the total consumption in industry m for a consumer j is the weighted sum of the consumption levels from the product of each individual firm i that the consumer is aware of at that time, c_{imjt}. The utility derived from this consumption is shifted by the ξ_{imj} idiosyncratic preference component, where a higher ξ_{imj} implies a better match between firm and consumer.

[11]The Gamma function is defined by $\Gamma : x \mapsto (x-1)!$. The constant $\overline{\Gamma}$ will simplify some algebraic expressions later on but carries no economic intuition.

Production technology and pricing Firm $i \in \mathcal{I}$ in industry $m \in [0, \boldsymbol{M}_t]$ produces output y_{imt} using the following Cobb-Douglas technology:

$$y_{imt} = z k_{imt}^{\alpha} l_{imt}^{1-\alpha}, \tag{3}$$

where $\alpha \in (0,1)$, k_{imt} is capital, l_{imt} is labor, and $z > 0$ is a common productivity component. Notice that, as firms have identical productivity, firm heterogeneity is driven entirely by the network of connections to consumers —in particular, the joint distribution of idiosyncratic preferences and awareness sets of those consumers who have that particular firm in their choice sets.

Each firm competes strategically with the other firms in the industry. We assume that firms engage in a repeated static Bertrand pricing game within their industry, simultaneously choosing their price as a best response to their competitors' prices, which they take as given.

Evolution of awareness Awareness sets $A_{mjt} \subseteq \mathcal{I}$ evolve endogenously and stochastically for each consumer-industry pair, $(m, j) \in [0, \boldsymbol{M}_t] \times [0, 1]$, as a result of the advertising choices described later on. In principle, this would require us to specify a law of motion for each consumer and industry in the continuum. However, as we will show in Proposition 2, in order to characterize the equilibrium, it suffices to keep track of a simpler object instead. Given the assumptions outlined in that proposition, we will argue that a sufficient statistic to calculate firm profits and prices is the distribution of the count of firms in consumer awareness sets. In anticipation of this result, we lay out assumptions regarding the law of motion of this distribution.

Let $a \in \mathbb{R}_+$ denote the age of an industry. Define the proportion of consumers aware of $n \in \{0, 1, \ldots, N\}$ firms at industry age a as $f_n(a)$, a probability mass function (pmf) with $f_n(a) \geq 0$ and $\sum_{n=0}^{N} f_n(a) = 1$, for any $a \geq 0$. Let us present these pmf's as a column vector $\vec{f}(a) \equiv [f_0(a), f_1(a), \ldots, f_N(a)]^\top \in \mathbb{R}_+^{N+1}$. The evolution of this distribution is assumed to follow a continuous-time Markov process.

Assumption 2 (Evolution of awareness)**.** *The law of motion of $\vec{f}(a)$ is:*

$$\partial_a \vec{f}(a) = \vec{f}(a) \cdot \mathcal{Q}, \tag{4}$$

given an initial condition $\vec{f}(0) \in \mathbb{R}_+^{N+1}$, where $\mathcal{Q}$ is the infinitesimal generator matrix:

$$\mathcal{Q} = \begin{bmatrix} -\theta & \theta & 0 & 0 & \ldots & 0 & 0 & 0 \\ \zeta & -\zeta - \frac{N-1}{N}\theta & \frac{N-1}{N}\theta & 0 & \ldots & 0 & 0 & 0 \\ 0 & \zeta & -\zeta - \frac{N-2}{N}\theta & \frac{N-2}{N}\theta & \ldots & 0 & 0 & 0 \\ 0 & 0 & \zeta & -\zeta - \frac{N-3}{N}\theta & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & \zeta & -\zeta - \frac{1}{N}\theta & \frac{1}{N}\theta \\ 0 & 0 & 0 & 0 & \ldots & 0 & \zeta & -\zeta \end{bmatrix}. \tag{5}$$

Assumption 2 states the following regarding network formation and destruction.[12] First, each consumer has an intensity $\theta > 0$ of becoming aware of a particular firm in the industry, so that when the consumer is aware of $n \leq N$ firms, the intensity with which she becomes aware of a new firm (i.e., of a firm that was not already in her awareness set) is $\frac{N-n}{N}\theta$. Hence, this rate affects the speed at which firms contact new customers. The contact rate θ is endogenous, common across firms within the industry, and chosen at the industry's inception. Second, we assume an intensity $\zeta \geq 0$ of losing links to an existing firm, i.e., the size of the awareness set shrinks by one connection at rate ζ, which we treat as exogenous.

Targeting As seen above, our assumption on preferences for the *population* of consumers is that they are Gumbel-distributed with location parameter equal to zero (Assumption 1). We further assume that, at any industry age $a \geq 0$, the preference shifters of consumers *who are aware* of a specific firm i are also distributed according to a Gumbel distribution from the firm's perspective, except with a location parameter $\widetilde{\mu}_i(a)$. Targeting allows picking the initial value of the location parameter $\widetilde{\mu}_{0,i} \equiv \widetilde{\mu}_i(0)$, thereby letting firms meet consumers who like their products earlier on. This choice is costly (a better initial match costs more), and made once-and-for-all at the time of inception of the industry. Thereafter, targeting evolves according to the law of motion stated in our next assumption. To state this assumption, let us define the set of awareness sets that contain a certain firm $i \in \mathcal{I}$ in industries of age a by:

$$\mathcal{A}_i(a) \equiv \big\{A \in \mathbb{A}(a) \big| i \in A\big\}, \tag{6}$$

where $\mathbb{A}(a) \equiv (A_j(a) : j \in [0,1])$ is the collection of all awareness sets in such an industry.

Assumption 3 (Evolution of targeting)**.** *At industry age a, targeting equals:*

$$\widetilde{\mu}_i(a) = \widetilde{\mu}_{0,i}(1 - s_i(a)), \tag{7}$$

where $s_i(a)$ denotes the degree of market saturation *for firm i at industry age a, defined by:*

$$s_i(a) \equiv \sum_{A \in \mathcal{A}_i(a)} \widehat{f}(a, A), \tag{8}$$

where $\widehat{f}(a, A)$ is the marginal density of awareness set A.

Equation (7) in Assumption 3 states that the targeting drops as the firm's network saturation increases, where saturation is defined as the proportion of awareness sets that contain the firm.[13] As we will see, in equilibrium, a firm's network will tend to become more saturated over time.

[12] It is worth pointing that nothing in our theory is tightly connected to this particular generator matrix, which we are taking as a model primitive. Richer versions, as well as limiting cases (e.g., $N \to +\infty$ and $\zeta = 0$, which relates to a pure counting process) could be used.

[13] Note that this assumption implicitly defines some primitives regarding the meeting probabilities of particular consumers with the firm (a higher $\widetilde{\mu}_{0,i}$ assigns more probability to meeting consumers with higher ξ_{imj}). This modeling choice allows the derivation of some closed-form solutions, as is common in the discrete-choice literature.

Therefore, as the industry matures, since more consumers become aware of the firm, the likelihood of meeting new consumers who have a higher preference for the firm compared to the population mean goes down. In the limit at which every consumer is aware of the firm's products ($s_i(a) \to 1$), it is no longer possible to have any systematic selection, and the distribution of preferences will match the unconditional distribution (i.e., $\widetilde{\mu}_i(a) \to 0$), at which point no more targeting is possible. This formulation captures the idea that targeting is most effective in the early stages of a product's lifecycle, and its effect vanishes over time as more consumers become aware of the firm's product.

Advertising costs When a new industry is created, the owner of the blueprint makes advertising choices. We solve the simple case where the blueprint owner establishes the common advertising choice before selling off, at fair market value, the N firms with a license to use the production technology. There are two advertising choices to be made: the contact rate, which governs the rate at which new consumers become aware of a firm, and targeting, which lets the firms in the industry reach out to those customers who have a better match with their product earlier than those who do not, on average. The advertising choices are paid upfront (at the inception of the industry) in units of the final good.[14] The advertising cost function is:

$$d(\theta_i, \mu_{0,i}) = \nu\theta_i^2 + \eta(\mu_{0,i} - 1)^2 \tag{9}$$

for each firm $i \in \{1, \ldots, N\}$, where $\nu > 0$ and $\eta > 0$ are parameters, and henceforth we use the change of variables $\mu_0 \equiv e^{\widetilde{\mu}_0}$.[15]

Investment in capital and product categories On top of being used for consumption and advertising, the single final good is also used for investment that increases the mass of industries, $\boldsymbol{M}_t$, and that which increases the stock of physical capital, $\boldsymbol{K}_t$. The technology to create new industries generates a Poisson arrival rate z_M of new products categories for each unit of the final good that is paid upfront. An industry free entry condition determines the measure $\boldsymbol{M}_t$ of categories in equilibrium. There is no entry or exit of firms within an industry over the industry's lifetime: all N firms are born when the industry is born, and all firms exit when the industry ceases to exist.

The investment technology for physical capital transforms one unit of the final good into one unit of physical capital. Physical capital depreciates at an instantaneous rate $\delta_K > 0$, and industries become obsolete at an exogenous rate $\delta_M > 0$. Therefore:

$$\partial_t \boldsymbol{K}_t = \boldsymbol{I}_t^K - \delta_K \boldsymbol{K}_t, \tag{10}$$

$$\partial_t \boldsymbol{M}_t = z_M \boldsymbol{I}_t^M - \delta_M \boldsymbol{M}_t, \tag{11}$$

[14] By assuming advertising technology is such that the optimal advertising decision is made at the entry stage, we substantially simplify the problem by making it an age-zero decision and avoiding strategic advertising considerations between firms within the industry. Future research can determine the extent to which strategic advertising decisions between operating firms affects consumer welfare.

[15] To characterize the equilibrium, it will be convenient to re-center the Gumbel draws for idiosyncratic consumer tastes to $\mu_0 \equiv e^{\widetilde{\mu}_0}$. Hence, we assume that the cost function scales with μ_0 directly rather than $\widetilde{\mu}_0$.

where $\boldsymbol{I}_t^K$ and $\boldsymbol{I}_t^M$ denote the respective investments, expressed in units of the final good.

2.2 Equilibrium

In this section, we solve for the Markov Perfect Equilibrium of the economy. We begin by characterizing the static and dynamic choices of the consumer, and then move on to describe firm choices and the evolution of industries. Having solved for these choices, we will show that the economy aggregates up to a representative-agent Neoclassical growth model with endogenous TFP and an endogenous number of product categories. Exploiting this finding will allow us to derive the equilibrium advertising choices.

2.2.1 Consumer Problem

The single final good of the economy is used for consumption, advertising, and investment in physical capital and industry creation. In the stationary equilibrium, a constant share of final output will be devoted to these various investments. In anticipation of this result, we solve the individual's intra-temporal allocation problem across industries and products as a static problem on quantities purchased, y_{imjt}, rather than consumed, c_{imjt}.[16]

Denote by Ω_{jt} the real income available to consumer j at time t, and by P_{jt} the price index of this individual's consumption bundle.[17] Taking her awareness sets $\left(A_{jmt} : m \in [0, \boldsymbol{M}_t]\right)$ as given, the objective of consumer $j \in [0,1]$ is to choose purchases y_{imjt} for each $i \in A_{mjt}$ and all $m \in [0, \boldsymbol{M}_t]$ to maximize static output:

$$Y_{jt} \equiv \left[\int_0^{\boldsymbol{M}_t} \left(\sum_{i \in A_{mjt}} \overline{\Gamma} e^{\sigma \xi_{imj}} y_{imjt} \right)^{\frac{\kappa-1}{\kappa}} \mathrm{d}m \right]^{\frac{\kappa}{\kappa-1}}, \tag{12}$$

subject to the budget constraint:

$$\int_0^{\boldsymbol{M}_t} \sum_{i \in A_{mjt}} \widehat{p}_{imt} y_{imjt} \mathrm{d}m \leq P_{jt} \Omega_{jt}, \tag{13}$$

where $\widehat{p}_{imt}$ is the (nominal) price of product (i, m) at time t. The following proposition describes the solution to this static resource allocation problem:

Proposition 1 (Product demand)**.** *Given awareness sets* $\left(A_{jmt} : m \in [0, \boldsymbol{M}_t]\right)$*, real income* Ω_{jt}*, and nominal prices* $\left(\{\widehat{p}_{imt}\}_{i \in A_{mjt}} : m \in [0, \boldsymbol{M}_t]\right)$*:*

1. (Extensive demand) *In industry* $m \in [0, \boldsymbol{M}_t]$*, consumer* j *purchases from firm* i *and from no other firm in her awareness set if and only if*

[16] The intertemporal allocation of resources is relegated to Section 2.2.3, after we show that the economy aggregates to a representative-household model.

[17] Both of these objects are solved in general equilibrium later (Proposition 5 and Appendix A.5), but for now it suffices to express them generically. Eventually, our assumptions will imply that all consumers have an identical price index.

$$\ln\left(\frac{\widehat{p}_{i'mt}}{\widehat{p}_{imt}}\right) > \sigma\left(\xi_{i'mj} - \xi_{imj}\right), \quad \forall i' \in A_{mjt}\backslash\{i\}. \tag{14}$$

2. (Intensive demand) *Suppose $i \in A_{mjt}$ satisfies condition (14). Then, the demand for firm i is:*

$$y^d_{imjt} = \overline{\Gamma}^{\kappa-1} e^{\sigma(\kappa-1)\xi_{imj}} p^{-\kappa}_{imjt} \Omega_{jt}, \tag{15}$$

where $p_{imjt} \equiv \widehat{p}_{imt}/P_{jt}$ denotes the real price, and

$$P_{jt} = \overline{\Gamma}^{-1}\left(\int_{\mathcal{M}_{jt}} \left(e^{-\sigma\xi_{i(m)mj}} \widehat{p}_{i(m)mt}\right)^{1-\kappa} dm\right)^{\frac{1}{1-\kappa}} \tag{16}$$

is the price index, where $\mathcal{M}_{jt} \equiv \{m : A_{mjt} \neq \emptyset\} \subseteq [0, \boldsymbol{M}_t]$ is the subset of industries for which consumer j is aware of at least one firm at time t and, for every $m \in \mathcal{M}_{jt}$, $i(m) \in A_{mjt}$ denotes the firm that satisfies equation (14).

Proof. See Appendix A.1.

Proposition 1 states that, for each industry m and at a given time t, the consumer demands from only one firm, almost surely, out of the $|A_{mjt}|$ firms in her awareness set. Particularly, in each m, the consumer demands the product $i(m)$ that satisfies condition (14). Equation (15) then provides the intensive demand for each such product, showing that the consumer's demand is increasing in preferences $\xi_{i(m)mj}$ and decreasing in the real price $p_{i(m)mjt}$, where $\kappa > 0$ is the price-elasticity of demand.

2.2.2 Firm Problem

At every instant, firms compete à la Bertrand, and then choose the input quantities needed to satisfy demand. As advertising choices are made once and for all at the industry's inception, we first consider the input and pricing decisions of firms taking advertising choices as given, from where we will be able to compute equilibrium profits and the age-zero value of a firm. Using the latter, we will then be able to solve the age-zero advertising choices.[18]

We start by solving for the demand faced by a particular firm. After having characterized the firm's demand curve (Proposition 2), we then present the solution to the pricing problem (Proposition 3), and only then close the firm's input and pricing problems by presenting the optimal choice of labor and capital inputs, given optimal prices (Proposition 4).

Throughout this section, we conjecture that both the price index and real income are identical across consumers: $P_{jt} = \boldsymbol{P}_t$ and $\Omega_{jt} = \boldsymbol{\Omega}_t$, $\forall j \in [0, 1]$. Under this conjecture, which we confirm in Proposition 5, note that real prices are constant as well, $p_{imjt} = p_{imt}$.

[18] Recall that the advertising choices are made by the owner of the industry's blueprint prior to licensing it to the entering firms, and not by the firms themselves (see Section 2.2.4).

Firm demand The firm faces demand from the subset of consumers whose awareness sets $A_{mj}(a)$ contain said firm's product. In addition, these consumers must choose the firm's product over any other products that they are currently aware of within the industry. Let $\vec{\hat{p}} \equiv [\hat{p}_{1,m}, \ldots, \hat{p}_{im}, \ldots, \hat{p}_{N,m}]^\top \in \mathbb{R}^N$ denote the vector of all nominal prices in the industry at a given time. Firm i maximizes profits taking the vector of competitors' nominal prices, $\vec{\hat{p}}_{-i} \equiv \vec{\hat{p}} \backslash \{\hat{p}_{im}\}$, as given. The total demand faced by firm i when setting price $\hat{p}$ is:

$$y_{im}(a, \hat{p}; \vec{\hat{p}}_{-i}) \equiv \int_0^1 \sum_{i \in A_{mj}(a)} y^d_{imj}(a) \mathbb{1}\left\{ \ln\left(\frac{\hat{p}_{i'm}(a)}{\hat{p}}\right) > \sigma(\xi_{i'mj} - \xi_{imj}) \middle| \forall i' \in A_{mj}(a) \backslash \{i\} \right\} \mathrm{d}j, \tag{17}$$

where $y^d_{imj}(a)$ is the intensive demand function that we found in equation (15), and $\mathbb{1}\{\cdot\}$ is an indicator function. Equation (17) states that a firm's demand is composed of the sum of demands from all those consumers that have the firm's product in their choice set. In turn, a firm is in consumer j's choice set if its product is in the consumer's awareness set and, in addition, the consumer decides to purchase it, as stated by the condition in the indicator function (coming from the extensive demand function from Proposition 1).

Computing the integral across consumers in equation (17) requires that the firm keeps track of the joint distribution of awareness sets and idiosyncratic preferences across those consumers whose awareness sets contain its product, a potentially complicated object. However, as our next proposition will show, firms need only keep track of the *size* of consumers' awareness sets. Because the evolution of awareness set sizes is independent of idiosyncratic preferences by virtue of Assumptions 1 and 2, the problem can be simplified greatly (for details, see the proof of Proposition 2 in the Appendix).

Another potential complication comes from the fact that the firm is assumed to offer a best-response to the prices of its $N-1$ competitors. These prices are themselves best responses, making the set of optimal strategies a potentially hard object to characterize. To make progress, we restrict our attention to *symmetric equilibria*, whereby a firm offers a best response to competitor prices and presumes that all such prices are equal to each other. As we will show shortly, an equilibrium with symmetric prices requires symmetry in targeting as well. Thus, we make the following assumption:

Assumption 4 (Common targeting)**.** *All of the firm's competitors have the same initial targeting, i.e.,* $\mu_{0,i'} = \mu_{0,-i}$*, for all* $i' \neq i$.

To be able to state firm demand, we must introduce one more piece of notation. Recalling that $f_n(a)$ is the proportion of consumers whose awareness sets have cardinality n at industry age a, we define the expectation of any function $g : \{1, \ldots, N\} \to \mathbb{R}_+$ as follows:

$$\mathbb{E}_a[g(\hat{n})] \equiv \sum_{n=1}^{N} \frac{f_n(a)}{1 - f_0(a)} g(n), \tag{18}$$

where $\hat{n} \equiv n | n \geq 1$. We are now ready to state our main result.

Proposition 2 (Firm demand with symmetric pricing strategies)**.** *Under Assumptions 1 to 4, and given real income $\mathbf{\Omega}_t$, a vector $\vec{p}_{-i} = \{p_{-i}, p_{-i}, \dots, p_{-i}\} \in \mathbb{R}^{N-1}$ of competitors'* real *prices, and a vector $\vec{\mu}_{-i} = \{\mu_{-i}, \mu_{-i}, \dots, \mu_{-i}\} \in \mathbb{R}_{+}^{N-1}$ of competitors' targeting values, the demand curve faced by a given firm with targeting $\mu(a)$ can be written as:*

$$y(a, p; p_{-i}, \mu_{-i}) = (1 - f_0(a))\mu(a)^{\sigma(\kappa-1)} p^{-\kappa} \frac{\mathbf{\Omega}_t}{N} q(a, p; p_{-i}, \mu_{-i}), \tag{19}$$

where $q(a, p; p_{-i}, \mu_{-i}) \equiv \mathbb{E}_a \left[\widehat{n} \left(1 + (\widehat{n} - 1) \frac{\mu_{-i}}{\mu(a)} \left(\frac{p_{-i}}{p} \right)^{-\frac{1}{\sigma}} \right)^{\sigma(\kappa-1)-1} \right]$.

Proof. See Appendix A.2.

Proposition 2 states that, in spite of the rich preference heterogeneity and time-varying consumer-specific awareness processes, the only relevant state coming from the consumer side that affects a firm's total demand, other than the level of targeting, is the *size* of the awareness sets. In particular, the *composition* of these sets is irrelevant for demand.

In a symmetric equilibrium with $p = p_{-i}$ and $\mu(a) = \mu_{-i}$, equation (19) becomes:

$$y(a, p) = \underbrace{(1 - f_0(a))}_{\text{Awareness}} \underbrace{\mu(a)^{\sigma(\kappa-1)}}_{\text{Targeting}} \underbrace{p^{-\kappa} \frac{\mathbf{\Omega}_t}{N}}_{\substack{\text{Downward-}\\ \text{sloping} \\ \text{demand}}} \underbrace{q(a)}_{\text{Sorting}}, \tag{20}$$

where, abusing notation slightly, we have defined $q(a) \equiv q(a, p; p, \mu(a))$. Using our definition from Proposition 2, note we can write $q(a)$ as follows:

$$q(a) = \mathbb{E}_a \left[\widehat{n}^{\sigma(\kappa-1)} \right]. \tag{21}$$

Equation (20) shows that demand is composed of four terms. First, a firm's demand is increasing in $\big(1 - f_0(a)\big)$, a term which we label "awareness" as it equals the proportion of consumers that are aware of at least one firm in the industry. Intuitively, the larger the share of consumers that are aware of the industry as a whole, the larger the demand of each specific firm within the industry, all else equal.

Second, a firm's demand is increasing in targeting, $\mu(a)$: consumers will demand more of a product if they have a stronger preference for it, all else equal. Recall that $\mu(a) = \mu_0 e^{1-s(a)}$, where μ_0 is the age-zero targeting choice and $s(a)$ is the network's saturation at age a. Under symmetry, all firms face the same market saturation which, using definition (8), can be written as:

$$s(a) = \frac{1}{N} \sum_{n=1}^{N} n f_n(a) = (1 - f_0(a)) \mathbb{E}_a \left[\frac{\widehat{n}}{N} \right]. \tag{22}$$

Thus, equation (20) shows that, through the initial targeting investment, a firm is able to increase the preference that a consumer has for its product at all points during the industry's lifecycle, thereby increasing demand for it. The impact of targeting on demand is measured by the elasticity $\sigma(\kappa-1)>0$, so that demand is more responsive to targeting if products are more substitutable and/or idiosyncratic preferences are more dispersed.

Third, demand is affected by $p^{-\kappa}\mathbf{\Omega}_t/N$, which is the typical component from Dixit-Stiglitz demand systems: consumers will increase their demand for a product, along the *intensive margin*, as their available income per product $\mathbf{\Omega}_t/N$ increases, and as price declines. Along this margin, the price-elasticity of demand coincides with the elasticity of substitution between product categories, κ.

Finally, demand is increasing in the term $q(a)\equiv\mathbb{E}_a\left[\widehat{n}^{\sigma(\kappa-1)}\right]$, which we label "sorting" as it relates to the average size of awareness sets, thereby capturing changes in demand along the *extensive margin*: when awareness sets are on average larger, firms face more intensive demand for their product due to the sorting of consumers towards products that yield a better match to their preferences.

This last component summarizes the endogenous components of market power and sorting in the model. To build intuition, suppose that a firm may face only one of two types of consumers: those whose awareness sets contain only the firm in question ($\widehat{n}=1$), and consumers whose sets contain two firms including the firm in question ($\widehat{n}=2$). For the former subset of consumers, the firm acts as a monopolist. For the latter subset, the firm acts as a duopolist. If the firm were to serve only the first kind of consumer, then $q(a)=1$ and consumers would only respond to changes in prices through the intensive margin component of demand, without intra-temporally switching to another firm. In contrast, if the firm were to serve only the second kind of consumer, then a change in the firm's price would affect the total quantity sold not only through the *intensive margin*, i.e., incumbent consumers of the firm moving along the downward-sloping intensive demand curve, but also through the *extensive margin*: as a result of the price change, the firm may end up losing consumers to, or gaining consumers from, the other firm. In a symmetric equilibrium all firms set the same price, so by equation (14) the firm would only serve consumers j for whom $\xi_{ij}>\xi_{-ij}$. But if the firm were to undercut firm $-i$'s price slightly (by, say, $\Delta>0$ log-points), then it would increase demand through better sorting: all those consumers j for whom $\xi_{ij}-\xi_{-ij}\in(-\Delta/\sigma,0)$, who would have otherwise purchased from firm $-i$, would now switch to firm i. In turn, this sorting of consumers into more preferred products would lead to a further increase in per-customer intensive demand, because of the factor $e^{\sigma(\kappa-1)\xi_{imj}}$ in equation (15).

Notice that the strength of these sorting forces in both extensive and intensive demand is governed by $\sigma>0$, the degree of preference heterogeneity across consumers: when $\sigma>0$ is higher, there is more scope for consumer sorting into better matches, and therefore more market power in the hands of firms to attract better matches.

Markups To understand how exactly firms exploit their market power along these margins, we next characterize the firm's pricing policy. First, note that because the firm's technology is Cobb-Douglas with constant returns to scale (equation (3)), the marginal cost is constant in the level of output and only a function of real input prices $\left(w_t,R_t^K\right)$, which the firm takes as given. In particular, the firm's input choice problem consists of minimizing total costs $\boldsymbol{TC}_t(y,a)$ to achieve a certain output level y subject to the available technology, or:

$$\boldsymbol{TC}_t(y,a) \equiv \min_{k(a),l(a)} \Big\{ \big(r_t + \delta_K\big)k(a) + w_t l(a) \quad \text{s.t. } y = zk(a)^{\alpha} l(a)^{1-\alpha} \Big\}, \tag{23}$$

where r_t, the real interest rate in the economy, holds $r_t = R_t^K - \delta_K$. As we show in Appendix A.4, the marginal cost is given by:

$$\boldsymbol{mc}_t \equiv \partial_y \boldsymbol{TC}_t(y,a) = \frac{1}{z}\left(\frac{r_t + \delta_K}{\alpha}\right)^{\alpha}\left(\frac{w_t}{1-\alpha}\right)^{1-\alpha}. \tag{24}$$

As the marginal cost is only a function of input prices and firm productivity, it is common across all firms and industries. This allows us to decouple the input choice problem from the pricing problem, so we analyze them separately.

Firms in an industry compete to maximize profits by playing a repeated Bertrand game, choosing a price policy taking as given the pricing decisions of the other firms in the industry. As the evolution of awareness sets is not directly affected by pricing decisions (Assumption 2), only indirectly through general-equilibrium effects, there are no dynamic incentives in this pricing game, and we solve for the case of a repeated static Nash equilibrium in pure strategies. Taking the marginal cost $\boldsymbol{mc}_t$ as given, we define a pure-strategy Bertrand Nash-equilibrium for each stage game at each period a as a vector of *real* prices $\vec{p}(a) \equiv \{p_i(a)\} \in \mathbb{R}_+^N$ such that, for every $i \in \mathcal{I}$, we have:

$$p_i(a) = \arg\max_{p \geq 0} \Big\{ \big(p - \boldsymbol{mc}_t\big) y\Big(a, p; \vec{p}_{-i}, \vec{\mu}_{-i}\Big) \Big\}, \tag{25}$$

where $y(\cdot)$ is given by equation (19). Then, for a symmetric equilibrium, we have:

Proposition 3 (Equilibrium markup)**.** *Taking $\boldsymbol{mc}_t$ as given, if a symmetric pure-strategy Bertrand Nash-equilibrium exists for N firms, then $p_i(a) = p(a) = \Lambda(a)\boldsymbol{mc}_t$, for all $i \in \mathcal{I}$, where the markup is:*

$$\Lambda(a) \equiv 1 + \frac{1}{\kappa - 1 - \mathcal{E}(a)}, \quad \textit{with } \mathcal{E}(a) \equiv -\frac{1-\sigma(\kappa-1)}{\sigma}\left[1 - \frac{\mathbb{E}_a\left[\widehat{n}^{\sigma(\kappa-1)-1}\right]}{\mathbb{E}_a\left[\widehat{n}^{\sigma(\kappa-1)}\right]}\right]. \tag{26}$$

Proof. See Appendix A.3.

Proposition 3 states that the firm sets a price markup $\Lambda(a)$ over the marginal cost, reflecting that firms exploit the market power they derive from the fact that their customers are potentially unaware of the existence of the firm's competitors. Intuitively, as all firms set the same price in equilibrium, the consumer chooses the product that yields the highest utility (by Proposition 1), i.e., firm i is chosen over firm $i' \in A_{mj}(a) \backslash \{i\}$ if, and only if, $\xi_{imj} > \xi_{i'mj}$. If a firm lowers its price, it can attract some consumers that would have otherwise chosen other firms in the awareness sets. A firm's markup therefore depends on the sensitivity of consumer switching to a change in price.[19]

[19]This idea is reminiscent of Paciello *et al.* (2019), where demand is also imperfectly elastic along the extensive margin.

This is encoded in $\mathcal{E}(a)$, a measure of the sensitivity of sorting to prices. Precisely, Appendix A.3 shows that $\mathcal{E}(a)$ equals the price-elasticity, evaluated at the symmetric equilibrium, of the sorting component of demand.

As $\mathcal{E}(a) \leq 0$, equation (26) shows that a lower price-elasticity of sorting (in absolute value) implies a higher markup.[20] In particular, $\mathcal{E}(a)$ decreases as consumers' awareness sets expand over time, meaning that the markup is monotonically decreasing in industry age, a.[21] At the early stages of the industry, consumers' awareness sets are small, which makes extensive-margin demand relatively inelastic and allows firms to exert more market power by setting higher markups. For instance, if awareness sets include just one firm, such firm acts as a monopolist for all of its consumers, and the markup is highest, at $\Lambda(a) = \frac{\kappa}{\kappa-1}$, corresponding to the markup from an otherwise frictionless monopolistically-competitive model.[22] As the industry matures, demand becomes more elastic as awareness sets expand and competition for customers intensifies. When the rate of link destruction is $\zeta = 0$, awareness sets eventually become fully connected (i.e., $\lim_{a\to+\infty} f_N(a) = 1$), and firms' networks fully saturated (i.e., $\lim_{a\to+\infty} s(a) = 1$), implying that $\lim_{a\to+\infty} \mathcal{E}(a) = -\frac{N-1}{N}\frac{1-\sigma(\kappa-1)}{\sigma}$. In this limit, markups converge to their lowest value, given by:

$$\lim_{a\to+\infty} \Lambda(a) = 1 + \sigma \left[1 - \frac{1-\sigma(\kappa-1)}{N}\right]^{-1}. \tag{27}$$

In this case, note that, even in fully mature industries, firms still set positive markups because they retain market power from consumer differentiation ($\sigma > 0$) and from the fact that firms are not atomistic and interact strategically ($N < +\infty$). This remains true in a limiting environment with atomistic firms and no strategic interaction ($N \to +\infty$), as in that case $\lim_{a\to+\infty} \mathcal{E}(a) = -\frac{1-\sigma(\kappa-1)}{\sigma}$ and therefore $\lim_{a\to+\infty} \Lambda(a) = 1 + \sigma > 1$.

Input choice We can now describe the optimal labor and capital input choices of the firm. The following proposition describes the solution in the symmetric pricing equilibrium:

Proposition 4 (Firm's input demands)**.** *Given real input prices (w_t, r_t), real income $\boldsymbol{\Omega}_t$ and marginal cost $\boldsymbol{mc}_t$, the firm's demand for labor and capital inputs is given by:*

$$l(a) = \frac{\boldsymbol{mc}_t}{w_t}(1-\alpha)y(a), \tag{28}$$

In that model, this occurs because consumers can search for alternative suppliers in response to price changes.

[20]The negative relationship between demand elasticities and markups is present in other models of variable markups, which obtain this relationship (albeit along the intensive margin of demand) in various settings, e.g., in models of oligopolistic competition à la Atkeson and Burstein (2008) or with Kimball preferences, e.g., Boar and Midrigan (2019). In our model, we achieve this relationship endogenously via the incompleteness of awareness sets.

[21]This property is unrelated to the decline in targeting $\mu(a)$ over the industry's life cycle, as $\mu(a)$ does not feature at all in the firm's markup. As we show in Appendix A.3, only the relative targeting between firms, $\mu_i(a)/\mu_{-i}(a)$, has an effect on markups. Under symmetric targeting, therefore, this effect disappears.

[22]Indeed, in this example, firm-level demand is perfectly inelastic. Since all awareness sets are composed of just one firm, $f_1(a) = 1$ and $f_n(a) = 0$, $\forall n \geq 2$, and therefore $\mathcal{E}(a) = \frac{1-\sigma(\kappa-1)}{\sigma}\left(\frac{\sum_{n=1}^N f_n(a) n^{\sigma(\kappa-1)-1}}{\sum_{n=1}^N f_n(a) n^{\sigma(\kappa-1)}} - 1\right) = 0$. Plugging $\mathcal{E}(a) = 0$ in equation (26), we obtain $\Lambda(a) = \frac{\kappa}{\kappa-1}$.

$$k(a) = \frac{\boldsymbol{mc}_t}{r_t + \delta_K} \alpha y(a), \tag{29}$$

where $y(a)$ is given by equation (19) evaluated at $p = p_{-i} = \Lambda(a)\boldsymbol{mc}_t$, that is:

$$y(a) = (1 - f_0(a))\mu(a)^{\sigma(\kappa-1)} \boldsymbol{mc}_t^{-\kappa} \Lambda(a)^{-\kappa} q(a) \frac{\boldsymbol{\Omega}_t}{N}. \tag{30}$$

Proof. See Appendix A.4.

An implication of this proposition is that the shares of firm sales accounted for by labor and capital input payments are both only functions of the firm's markup:

$$w_t \frac{l(a)}{p(a)y(a)} = (1-\alpha)\Lambda(a)^{-1} \quad \text{and} \quad (r_t + \delta_K)\frac{k(a)}{p(a)y(a)} = \alpha\Lambda(a)^{-1}. \tag{31}$$

This means, in turn, that we can write total firm profits as:

$$\pi(a) \equiv p(a)y(a) - \boldsymbol{TC}_t(a, y(a)) = p(a)y(a)\left(1 - \Lambda(a)^{-1}\right). \tag{32}$$

Using equation (30) and $p(a) = \Lambda(a)\boldsymbol{mc}_t$, equation (32) allows us to write the total period profits of the firm as a function of the marginal cost, the markup, and the different components of firm demand identified above.

2.2.3 Aggregation

Having found the optimal choices of consumers and firms, and before deriving advertising choices, we next characterize the dynamics of industries and of the aggregate economy. To this end, we first show that, in spite of the rich household heterogeneity, the model aggregates to a representative-agent Neoclassical growth economy in which limited awareness at the microeconomic level is embedded in wedges in aggregate TFP at the macroeconomic level.

To arrive at this result, we must aggregate up from the industry level. Let us denote by $\Phi_t(a)$ the cumulative density function (cdf) of the age distribution of industries as of time t, with $\Phi_t(0) = 0$ and $\lim_{a\to+\infty} \Phi_t(a) = 1$, for all $t \in \mathbb{R}_+$. Let $\phi_t(a)$ be the probability density function (pdf) associated with this distribution. Since the instantaneous rate of industry creation (i.e., the flow of new industries per unit of time) equals $z_M \boldsymbol{I}_t^M$, the law of motion for the age distribution of products is given by the Kolmogorov Forward Equation:

$$\partial_t \widehat{\Phi}_t(a) = -\underbrace{\partial_a \widehat{\Phi}_t(a)}_{\substack{\text{Industry} \\ \text{aging}}} - \underbrace{\delta_M \widehat{\Phi}_t(a)}_{\text{Obsolescence}} + \underbrace{z_M \boldsymbol{I}_t^M}_{\substack{\text{Industry} \\ \text{creation}}}, \tag{33}$$

where we have defined $\widehat{\Phi}_t(a) \equiv \boldsymbol{M}_t \Phi_t(a)$. Moreover, define the total labor demand in an industry of age a by $L(a) \equiv Nl(a)$, with $l(a)$ given by equation (28). As there is a unit supply of labor and a measure $\boldsymbol{M}_t$ of industries, the total labor demand in the economy is:

$$\boldsymbol{L}_t \equiv \boldsymbol{M}_t \int_0^{+\infty} L(a)\phi_t(a)\mathrm{d}a. \tag{34}$$

By labor market clearing, $\boldsymbol{L}_t = 1$. We may also define the aggregate stock of capital by $\boldsymbol{K}_t \equiv \boldsymbol{M}_t \int_0^{+\infty} K(a)\phi_t(a)\mathrm{d}a$, where $K(a) \equiv Nk(a)$ is the industry's demand for capital and $k(a)$ is firm-level capital demand, given by equation (29). Likewise, we define aggregate profits by $\boldsymbol{\Pi}_t \equiv \boldsymbol{M}_t \int_0^{+\infty} \Pi(a)\phi_t(a)\mathrm{d}a$, where $\Pi(a) \equiv N\pi(a)$ are industry-level profits and $\pi(a)$ are firm-level profits, given by equation (32). Finally, define the following aggregate objects, whose interpretation we provide later on:

$$\boldsymbol{Q}_t \equiv \left(\int_0^{+\infty} (1 - f_0(a))\Lambda(a)^{1-\kappa}\mu(a)^{\sigma(\kappa-1)}q(a)\phi_t(a)\mathrm{d}a\right)^{\frac{1}{\kappa-1}}, \tag{35}$$

$$\boldsymbol{B}_t \equiv \frac{\int_0^{+\infty} (1 - f_0(a))\Lambda(a)^{-\kappa}\mu(a)^{\sigma(\kappa-1)}q(a)\phi_t(a)\mathrm{d}a}{\int_0^{+\infty} (1 - f_0(a))\Lambda(a)^{1-\kappa}\mu(a)^{\sigma(\kappa-1)}q(a)\phi_t(a)\mathrm{d}a}. \tag{36}$$

With these definitions, we arrive at our main aggregation result:

Proposition 5 (Aggregation)**.** *Under Assumptions 1 to 4, the economy aggregates to a representative-agent Neoclassical growth model. In particular:*

1. *The price index and real income are identical across consumers: $P_{jt} = \boldsymbol{P}_t$ and $\Omega_{jt} = \boldsymbol{\Omega}_t$, $\forall j \in [0,1]$. Moreover, real income equals total output from the composite good defined in equation (12), i.e., $\boldsymbol{\Omega}_t = Y_{jt} = \boldsymbol{Y}_t$, and it can be expressed as follows:*

$$\boldsymbol{Y}_t = \boldsymbol{Z}_t \boldsymbol{K}_t^{\alpha} \boldsymbol{L}_t^{1-\alpha}, \tag{37}$$

where $\boldsymbol{Z}_t$ is aggregate TFP, given by:

$$\boldsymbol{Z}_t \equiv z\boldsymbol{M}_t^{\frac{1}{\kappa-1}} \boldsymbol{Q}_t \boldsymbol{B}_t^{-1}. \tag{38}$$

2. *Real income is exhausted by labor, capital and profit income payments, i.e.,*

$$\boldsymbol{Y}_t = w_t \boldsymbol{L}_t + (r_t + \delta_K)\boldsymbol{K}_t + \boldsymbol{\Pi}_t, \tag{39}$$

with the following income shares: $\frac{w_t \boldsymbol{L}_t}{\boldsymbol{Y}_t} = (1-\alpha)\boldsymbol{B}_t$, $\frac{(r_t+\delta_K)\boldsymbol{K}_t}{\boldsymbol{Y}_t} = \alpha \boldsymbol{B}_t$, and $\frac{\boldsymbol{\Pi}_t}{\boldsymbol{Y}_t} = 1 - \boldsymbol{B}_t$.

Proof. See Appendix A.5.

This proposition states that the economy collapses to that of a representative-household Neoclassical growth model, in which the micro-level frictions in the form of slow-moving product awareness imply distortions to aggregate TFP.

Aggregate TFP, defined in equation (38), has different components. First, it scales with the physical productivity of firms, z. Second, it increases with $\boldsymbol{M}_t^{\frac{1}{\kappa-1}}$, the total number of product categories in the economy: as in standard expanding-variety growth models, the introduction of new varieties boosts aggregate productivity. Finally, there is an endogenous distortion $\boldsymbol{Q}_t\boldsymbol{B}_t^{-1}$ to TFP, which fully summarizes the aggregate effects of limited awareness in equilibrium. In this distortion, $\boldsymbol{Q}_t$ (defined in equation (35)) is an aggregate measure of *match quality*, as it incorporates the effects of both the sorting and the targeting components of demand, $q(a)$ and $\mu(a)$. On the other hand, $\boldsymbol{B}_t \leq 1$ (defined in equation (36)), adjusts this quality index for markup distortions induced by the limited awareness process.[23] Specifically, the *distortion-adjusted quality* measure $\boldsymbol{Q}_t\boldsymbol{B}_t^{-1}$ shows up as a wedge to TFP because firms exploit the fact that their customers are unaware of direct competitors to wield market power over them.

More precisely, as seen in equations (35)-(36), the size of the $\boldsymbol{Q}_t\boldsymbol{B}_t^{-1}$ wedge depends on (i) the degree of connectedness of awareness sets across different industries, summarized by the sorting component of demand, $q(a)$; (ii) the level of targeting in advertising, $\mu(a)$; (iii) the dispersion in markups across industries, $\Lambda(a)$; and (iv) the share of consumers that remain unaware of firms in the industry, $f_0(a)$. Note that, as in other models of variable markups with non-constant demand price-elasticities, only markup dispersion, and not the level of markups, has an aggregate distortionary effect. Indeed, if there was no dispersion in markups across industries, so that $\Lambda(a) = \Lambda$ for all $a > 0$, then the term $\boldsymbol{Q}_t\boldsymbol{B}_t^{-1}$ would be independent of Λ, and only reflect sorting- and targeting-induced quality. In contrast, sorting $q(a)$, targeting $\mu(a)$, and overall awareness $f_0(a)$, all have both level and dispersion effects on aggregate TFP.

The evolution of the TFP wedge over time is, in turn, driven by the evolving age distribution of firms in an industry, $\Phi_t(a)$, described in equation (33), and hence it depends on the underlying awareness process summarized by the transition matrix, $\mathcal{Q}$.

2.2.4 Advertising and Dynamic Resource Allocation

We are ready to solve for the age-zero advertising choices and the intertemporal consumption-saving decisions. As the model aggregates to a representative-household economy, the intertemporal allocation of total output $\boldsymbol{Y}_t$ between consumption and investment is akin to that of the Neoclassical growth model. In fact, as all input markets are perfectly competitive, and frictions in the output market are fully embedded into a single aggregate TFP wedge, the intertemporal allocation of aggregate resources can be characterized from the problem of a representative household making consumption, advertising, and investment decisions from the composite good $\boldsymbol{Y}_t$ (whose price we normalize to $\boldsymbol{P}_t = 1$).

This household invests in and rents away physical capital $\boldsymbol{K}_t$ to the firms at the rental rate $r_t + \delta_K$, supplies labor inelastically in exchange for the equilibrium wage w_t, invests into the creation of new industries $\boldsymbol{I}_t^M$, and accumulates wealth $\boldsymbol{A}_t$ at the interest rate r_t. The household trades in firm

[23] In fact, $\boldsymbol{B}_t$ can be thought of as an aggregate measure of the degree of market power in the economy, since $1 - \boldsymbol{B}_t$ equals the share of GDP that goes to firm profits by Proposition 5.

shares, so total financial wealth is given by:

$$\boldsymbol{A}_t = \boldsymbol{M}_t \int_0^{+\infty} V_t(a)\phi_t(a)\mathrm{d}a, \tag{40}$$

where $V_t(a)$ denotes the value of an industry of age a at time t. This value is computed as the present discounted value of the whole future stream of profits:

$$V_t(a) \equiv \int_t^{+\infty} e^{-\int_t^s (r_\tau+\delta_M)\mathrm{d}\tau}\Pi_s(a+s-t)\mathrm{d}s. \tag{41}$$

We start with the endogenous advertising choices. When a new industry is created, the owner of the blueprints to be used by the N firms chooses a common contact rate $\widehat{\theta}$ and targeting $\widehat{\mu}_0$ for all the firms in the industry.[24] After this choice, the blueprints are sold off to the household at fair value. The advertising choice $(\widehat{\theta}, \widehat{\mu}_0)$ maximizes the value of an industry at age zero net of advertising costs:

$$\boldsymbol{V}_t^0 \equiv \max_{\widehat{\theta},\widehat{\mu}_0} \left\{ V_t(0) - Nd(\widehat{\theta}, \widehat{\mu}_0) \right\}, \tag{42}$$

where recall that $d(\theta, \mu_0) \equiv \nu\theta^2 + \eta(\mu_0 - 1)^2$ is the firm-level advertising cost function defined in equation (9). Note $V_t(0)$ is an implicit function of advertising choices, as these affect the evolution of awareness sets (by Assumption 2) and directly impact firm demand by affecting match quality, i.e., the combined effects of sorting and targeting (by Proposition 2). The solution of problem (42), denoted $(\widehat{\theta}_t^*, \widehat{\mu}_{0,t}^*)$, gives the optimal choices as:

$$\widehat{\theta}_t^* = \frac{1}{2N\nu}\left[\partial_{\widehat{\theta}} V_t(0)\right] \qquad \text{and} \qquad \widehat{\mu}_{0,t}^* = 1 + \frac{1}{2N\eta}\left[\partial_{\widehat{\mu}_0} V_t(0)\right]. \tag{43}$$

In a stationary equilibrium, the contact rate and targeting are constant across industries and time, so $\widehat{\theta}_t^* = \theta$ and $\widehat{\mu}_{0,t}^* = \mu_0$.

Next, we solve for the dynamic problem of the household. Given initial conditions $\boldsymbol{A}_0$, $\boldsymbol{K}_0$, $\boldsymbol{M}_0$, and $\Phi_0(a)$, the problem is:

$$\max_{(\boldsymbol{C}_t, \boldsymbol{I}_t^K, \boldsymbol{I}_t^M \geq 0)_{t\in\mathbb{R}_+}} \int_0^{+\infty} e^{-\rho t}\frac{\boldsymbol{C}_t^{1-\gamma}}{1-\gamma}\mathrm{d}t, \tag{44a}$$

$$\text{subject to} \qquad \partial_t \boldsymbol{A}_t = r_t\boldsymbol{A}_t + w_t + (r_t+\delta_K)\boldsymbol{K}_t - \boldsymbol{C}_t - \boldsymbol{I}_t^K - \boldsymbol{I}_t^M + z_M\boldsymbol{I}_t^M\boldsymbol{V}_t^0, \tag{44b}$$

$$\partial_t \boldsymbol{K}_t = \boldsymbol{I}_t^K - \delta_K\boldsymbol{K}_t. \tag{44c}$$

[24] Note that we assume the owner of the blueprints in the new industry must choose the same $\widehat{\theta}_i = \widehat{\theta}$ and $\widehat{\mu}_{0,i} = \widehat{\mu}_0$ for all firms i. This assumption prevents the owner from reducing the effective number of firms in the industry through setting $\widehat{\theta}_i = 0$ and $\widehat{\mu}_{0,i} = 0$ for some firms, and ensuring they never contact any consumers, effectively removing the firm from the set of competing firms and thereby boosting the value of remaining firms with positive $\widehat{\theta}$. Another way through which we could achieve the same outcome in equilibrium rather than an assumption would be to let the firms make the advertising decisions themselves after they are sold off by the initial owner, but this would violate Assumption 2 and render the derivation of closed-form solutions infeasible. Calculating such a competitive advertising equilibrium through computation is a promising direction for future research.

The flow budget constraint (44b) states that the change in total assets comes from the returns on outstanding assets, income from supplying labor and renting capital to the firms, and the return from creating new industries, net of consumption and investment expenditures. The household's problem delivers the standard consumption Euler equation:

Proposition 6 (Euler equation)**.** *The law of motion of aggregate consumption is:*

$$\frac{\partial_t \boldsymbol{C}_t}{\boldsymbol{C}_t} = \frac{r_t - \rho}{\gamma}. \tag{45}$$

Proof. See Appendix A.6.

It remains to describe the investment choices in industry creation. This choice is made ex-ante and its costs are paid upfront. Since each unit of the final good produces a Poisson arrival rate z_M of new industries, and an industry has value $\boldsymbol{V}_t^0$, the household will invest up to the point at which the return from its investment equals the upfront cost. This leads to the following industry free entry condition:

$$\forall t \in \mathbb{R}_+ : \quad z_M \boldsymbol{V}_t^0 \leq 1 \quad \text{with equality if, and only if, } \boldsymbol{I}_t^M > 0. \tag{46}$$

We focus on an equilibrium with a positive entry of new industries, which means that $\boldsymbol{I}_t^M > 0$ and $\boldsymbol{V}_t^0 = 1/z_M$ in equilibrium.

2.2.5 Closing the Model

To close the model, it remains to impose market clearing in the goods market. The resource constraint of the economy is:

$$\boldsymbol{Y}_t = \boldsymbol{C}_t + \boldsymbol{I}_t^K + \boldsymbol{I}_t^M + \boldsymbol{D}_t, \tag{47}$$

where $\boldsymbol{D}_t$ denotes aggregate expenditure in advertising, defined by:

$$\boldsymbol{D}_t \equiv z_M \boldsymbol{I}_t^M N d(\theta, \mu_0). \tag{48}$$

Total advertising expenditures are equal to the upfront per-firm costs $d(\theta, \mu_0)$, times the measure of entering firms at time t (when a new industry is created), equal to $z_M \boldsymbol{I}_t^M N$.

Finally, in a stationary equilibrium, the age distribution of industries is time-invariant, $\Phi_t(a) = \Phi(a)$, and economic aggregates are constant over time. In this case, we can obtain the following closed-form solution for the invariant distribution of industries:

Proposition 7 (Stationary age distribution)**.** *The invariant age distribution is given by:*

$$\Phi(a) = 1 - e^{-\delta_M a}. \tag{49}$$

Proof. See Appendix A.7.

3 Quantitative Analysis

Over the last couple of decades, the advent and rise of digital advertising has dramatically changed the advertising landscape. One notable difference between digital and more traditional advertising media resides in the degree to which digital advertising can be targeted to consumers who are more likely to purchase the product. In this section, we calibrate the model to analyze how targeting has affected consumer product awareness, match quality, firm dynamics, markups, and welfare over time.

3.1 Calibration Strategy

We separately calibrate the model twice, for the years 2005 (*early calibration*) and 2014 (*late calibration*). Over this period, the share of internet advertising to total advertising increased significantly, from 6 to 30 percent.[25] Aggregate advertising spending, by contrast, remained relatively constant as a share of U.S. GDP at around 2.2 percent (see, e.g., Greenwood *et al.* (2021)). Our purpose in this exercise is to understand the degree to which this rise in digital advertising may have affected industry and aggregate dynamics, as well as markups and welfare.

For each calibration, we have 13 parameters. We externally calibrate several parameters commonly encountered in macroeconomic models. The rest of the parameters, which are most closely related to advertising, are internally calibrated.

Externally calibrated We set the following parameters externally, which are kept constant across the two calibrations. The model's period is one year, and the time discount rate is $\rho = 0.04$. The risk aversion parameter is set to $\gamma = 2$, consistent with an elasticity of intertemporal substitution of 0.5, documented for the U.S. by Havranek, Horvath, Irsova and Rusnak (2015). The capital share as a fraction of non-profit income is set to $\alpha = 1/3$. The cross-industry elasticity of substitution is set to $\kappa = 2$, which is consistent with the range of estimates calculated in Oberfield and Raval (2021) for highly disaggregated industries. This choice for κ puts an upper bound on net markups at 100 percent, corresponding to the monopolistically competitive markup (recall our discussion following Proposition 3). We normalize the common firm-level productivity to $z = 1$, which is without loss of generality. To isolate the endogenous role of advertising on the awareness process, we set the exogenous rate of losing connections to $\zeta = 0$. We choose a relatively large number of firms per product category, $N = 10$.[26] Physical capital depreciation is set to $\delta_K = 0.069$, as in Celik, Tian and Wang (2022), who in turn use data from the U.S. NIPA tables. Finally, we interpret the destruction rate δ_M as the rate at which product categories exit the economy. Broda and Weinstein (2010) find that, at the manufacturer level, this rate is about 9 percent per year. Interpreting the industries in

[25] Obtained from Statista, using data from Zenith's Advertising Expenditure Forecasts (December 2021 report): https://www.statista.com/statistics/429036/advertising-expenditure-in-north-america/.

[26] As awareness sets in a product category are typically found to be very small (see our review of the literature in the introduction), we view our choice of $N = 10$ as being quite conservative.

our model as being composed of firms that produce goods within the same product category, we set $\delta_M = 0.09$.

Internally calibrated There are four remaining parameters: the cost of new industry creation (z_M), the degree of product differentiation in preferences (σ), and the advertising cost scale parameters for contacting (ν) and targeting (η). We calibrate these parameters internally. For each of the two calibrations, we choose values for the four parameters that minimize the distance between model-generated moments and their empirical counterparts.

For both calibrations, we normalize the mass of industries to $\boldsymbol{M} = 1$, which allows us to pin down the value for the cost of creating new industries, z_M. We then target the ratio of advertising expenditures to GDP, using data from Greenwood *et al.* (2021); the sales-weighted average markup, using estimates from De Loecker, Eeckhout and Unger (2020); and a measure of the effectiveness of digital advertising. The latter measure is based on empirical evidence on the return to targeting reported in Farahat and Bailey (2012). That study finds, using a natural field experiment from ads on the Yahoo! homepage, that targeting increases the click-through rate for brands by 79 percent on average. As the share of digital advertising expenses in total advertising expenditures in the data increased from around 6 percent to 30 percent between 2005 and 2014, we weight the return to targeting by the share of digital advertising to be able to compute the average return to targeting. By this measure, the return to targeting goes from 0.048 in 2005 to 0.213 in 2014, nearly a five-fold increase. In the model, we measure the return to targeting by computing the expected increase in a given firm's sales under the assumption that every other firm in the industry does not use targeting at all, i.e., that all other firms in the industry choose $\mu_0 = 1$, which is intended to replicate the experiment conducted in Farahat and Bailey (2012) using our model.

For the late calibration (2014), the internally calibrated parameters are estimated against the same set of moments. In addition, we also make sure that the growth rate of real GDP per capita between the two calibrations is in line with what is observed in the data.[27]

3.2 Calibration Results

Table 1 reports the results for the early and late calibrations in terms of model fit, and provides the corresponding parameter values. The model matches all moments closely, in particular, the stable share of advertising in GDP over time combined with the increase in the degree of advertising targeting.

Regarding parameter values, our calibrations predict little change in consumer preference heterogeneity, from $\sigma = 0.4183$ in 2005 to $\sigma = 0.4099$ in 2014, due to the fact that the average markup changed very little over this period. Instead, the large changes in the composition of advertising seen in the data are being captured mostly through changes in the advertising technology. Importantly, both the cost of contacting and the cost of targeting are lower in the late calibrations.

[27] Data on GDP per capita is obtained from the St. Louis Federal Reserve Bank's FRED database, and is available at https://fred.stlouisfed.org/series/A939RX0Q048SBEA#0.

Table 1: Full set of parameters and model fit

Parameter		**Value**	**Source/Target**		
A. Externally identified (2005 and 2014)					
Number of firms per industry	N	10			
Firm-level productivity	z	1			
Connection destruction rate	ζ	0			
Time discount rate	ρ	0.04	4% annual interest rate		
Cross-industry elasticity of substitution	κ	2	Oberfield and Raval (2021)		
Capital share of non-profit income	α	0.33	Capital share of non-profit income		
Coefficient of relative risk aversion	γ	2	Havranek *et al.* (2015)		
Capital depreciation	δ_K	0.069	Celik *et al.* (2022) and U.S. NIPA tables		
Product destruction rate	δ_M	0.09	Broda and Weinstein (2010)		
Parameter		**Value**	**Moment**	**Data**	**Model**
B. Internally identified (2005)					
Product differentiation	σ	0.4183	Sales-weighted average markup	0.4674	0.4658
Industry creation efficiency	z_M	0.1059	Mass of industries (normalization)	1.0000	1.0000
Contact rate cost	ν	0.0267	Advertising share of GDP	0.0220	0.0220
Targeting cost	η	0.2527	Return to targeting	0.0482	0.0482
C. Internally identified (2014)					
Product differentiation	σ	0.4099	Sales-weighted average markup	0.4850	0.4603
Industry creation efficiency	z_M	0.0999	Mass of industries (normalization)	1.0000	1.0000
Contact rate cost	ν	0.0229	Advertising share of GDP	0.0224	0.0224
Targeting cost	η	0.0352	Return to targeting	0.2129	0.2129
			Real GPD per capita growth	0.0523	0.0524

Notes: The model period is one year. Panel A reports the externally calibrated parameters. Panel B reports parameter values and model fit for the early calibration, corresponding to data moments from 2005. Panel C reports results for the late calibration, corresponding to 2014.

The decrease in the cost of targeting is predicted to have been large: the scale parameter decreases from $\eta = 0.2527$ to $\eta = 0.0352$, a 86 percent decline, between the 2005 and 2014 calibrated economies. The contact rate cost decreases as well but to a lesser extent, from $\nu = 0.0267$ to $\nu = 0.0229$, a 14 percent decline.

Table 2 reports the baseline results for the early and late calibrations on a number of selected variables, including various advertising outcomes and variables related to markups and the sources of aggregate expenditure. As a result of the changes in ν and η between the two calibrations, initial targeting μ_0 goes up strongly between the early and late calibrations, from $\mu_0 = 1.230$ to $\mu_0 = 2.088$, a 70 percent increase, owing to the fact that the return on targeting has increased over this period. Consequently, average targeting, computed by $\boldsymbol{\mu} \equiv \int_0^{+\infty} \mu(a)^{\sigma(\kappa-1)}\phi_t(a)\mathrm{d}a$, goes up as well, from $\boldsymbol{\mu} = 0.023$ to $\boldsymbol{\mu} = 0.101$, a nearly 5-fold increase. On the other hand, the contact rate decreases slightly, from $\theta = 1.924$ to $\theta = 1.853$, a 3.8 percent decline, between the two periods. Therefore, our calibrations predict that firms have reduced the probability of contacting new customers over time, but have now become better at targeting customers with greater preference for their products. As a

Table 2: Baseline calibration results

		(1) Early calibration	(2) Late calibration
A. Advertising and markups			
Contact rate	θ	1.924	1.853
Targeting rate	μ_0	1.230	2.088
Average targeting	$\boldsymbol{\mu}$	0.023	0.101
Average return to targeting		0.048	0.213
Average sales-weighted markup		0.466	0.460
Firm value (without adv. cost)	$V(0)$	1.057	1.121
B. Shares of GDP			
Consumption share	$\boldsymbol{C}/\boldsymbol{Y}$	0.649	0.647
Advertising share	$\boldsymbol{D}/\boldsymbol{Y}$	0.022	0.022
Industry creation investment share	$\boldsymbol{I}^M/\boldsymbol{Y}$	0.185	0.186
Capital investment share	$\boldsymbol{I}^K/\boldsymbol{Y}$	0.144	0.144
Profit share	$\boldsymbol{\Pi}/\boldsymbol{Y}$	0.318	0.315
C. Economic aggregates			
Mass of industries	$\boldsymbol{M}$	1.000	1.000
Wage	w	2.087	2.205
Consumption level	$\boldsymbol{C}$	2.977	3.123
Match quality	$\boldsymbol{Q}$	1.474	1.529
Distortion-adjusted quality	$\boldsymbol{QB}^{-1}$	2.161	2.233
Output level	$\boldsymbol{Y}$	4.589	4.829
Aggregate TFP	$\boldsymbol{Z}$	2.161	2.233

Notes: Results from our calibrations on selected equilibrium variables. Column (1) reports the baseline results for the early calibration (2005). Column (2) reports results for the late calibration (2014).

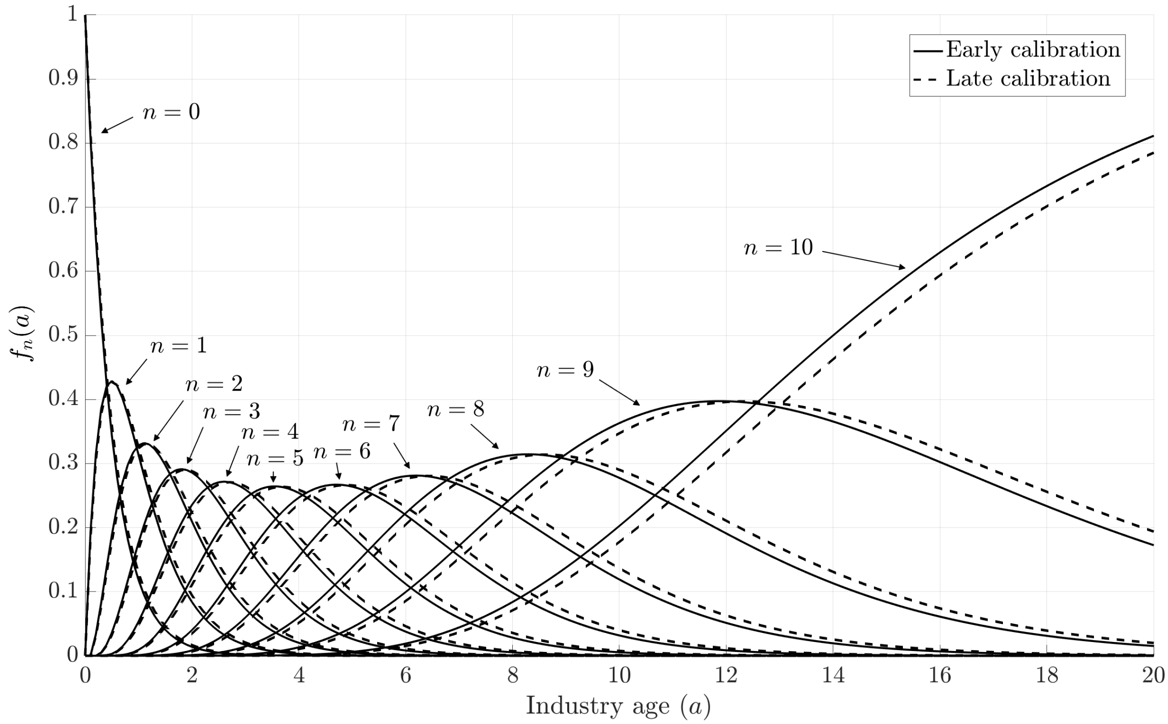
result of these changes, firm value increases from 2005 to 2014, by 6.05 percent, and both overall output and overall consumption go up, by 5.6 percent and 4.9 percent, respectively, a substantial increase in consumption-equivalent welfare. Correspondingly, the level of aggregate distortions is lower in the late economy: recalling that $\boldsymbol{QB}^{-1}$ is the endogenous component of aggregate TFP, we obtain that $\boldsymbol{QB}^{-1}$ increases by 3.3 percent, from $\boldsymbol{QB}^{-1} = 2.161$ in 2005 to $\boldsymbol{QB}^{-1} = 2.233$ in 2014, as both aggregate match quality ($\boldsymbol{Q}$) increases, and aggregate distortions from market power ($\boldsymbol{B}^{-1}$) decrease.

3.3 Effects on Industry Dynamics

The aggregate effects discussed above reflect underlying changes in the dynamics of industries, to which we move next.

Figure 1 shows the evolution of product awareness over time within an industry. Each solid line (respectively, dashed line) represents the share of consumers aware of a certain number of products, $f_n(a)$, in the early (respectively, late) calibration, as a function of the age of the industry. Upon industry creation ($a = 0$), no consumer is aware of any product, so $f_0(0) = 1$. As time goes by and

Figure 1: Proportion of consumers aware of n firms, $f_n(a)$.



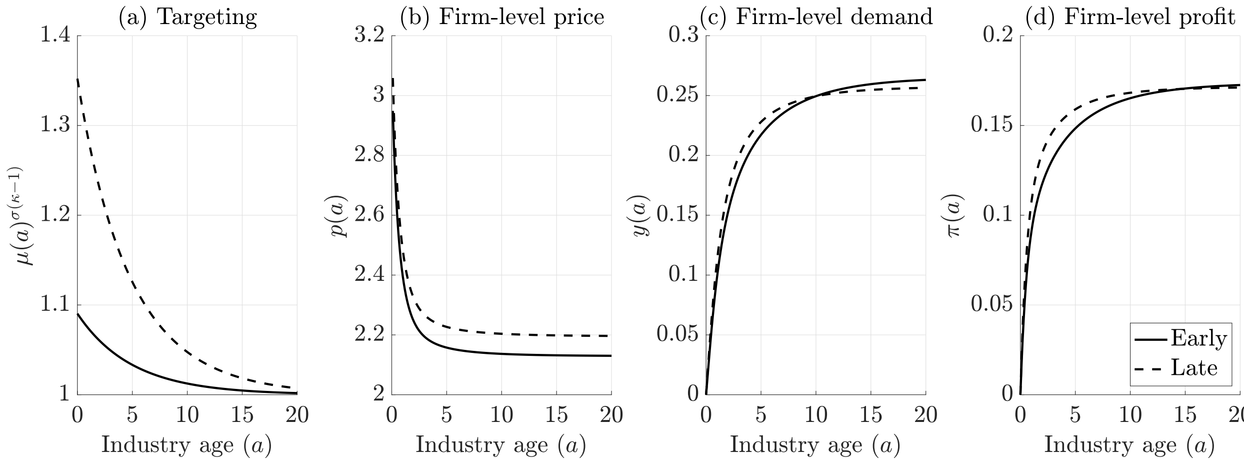
Notes: This figure plots the proportion of consumers that are aware of $n = 0, 1, \ldots, N$ firms over industry age a, for the early calibration (solid lines) and the late calibration (dashed lines).

the industry ages, consumers become gradually aware of the existence of products and their product awareness sets expand. In the long run, consumers come to learn about all the products in the industry, and $\lim_{a\to+\infty} f_N(a) = 1$ in both calibrations.[28] Comparing early and late calibrations, we notice that consumers in the early period get to know more products earlier on, as the contact rate (θ) is higher.

Even though consumers' awareness sets grow more slowly in the late calibration relative to the early one, targeting is higher, especially in the early stages of the industry when sorting is low and firms retain a relatively high degree of market power (panel (a) in Figure 2). As a result, firms charge higher prices in the late calibration (panel (b)). Firm-level demand (panel (c)) reflects the conflicting effects of both higher prices and higher targeting. In the early stages of the industry, the increase in targeting dominates over the price effect, and demand is higher in the late calibration. As consumers become aware of all the firms in the industry over time, targeting converges in both the early and late calibrations, and the price effect eventually dominates, resulting in higher demand for firms in the later stages of the industry's life cycle (specifically, after the tenth year). A similar pattern is observed for profits, in panel (d) of Figure 2, though in that case differences reflect not only demand changes but also markups. Once again, the firm-level profits are higher for the late

[28]This asymptotic result owes to (i) setting $\zeta = 0$, and (ii) to the relatively high industry destruction rate at $\delta_M = 9\%$, so that most industries are destroyed long before $f_n(a)$ tends to 10.

Figure 2: Firm-level outcomes by industry age.

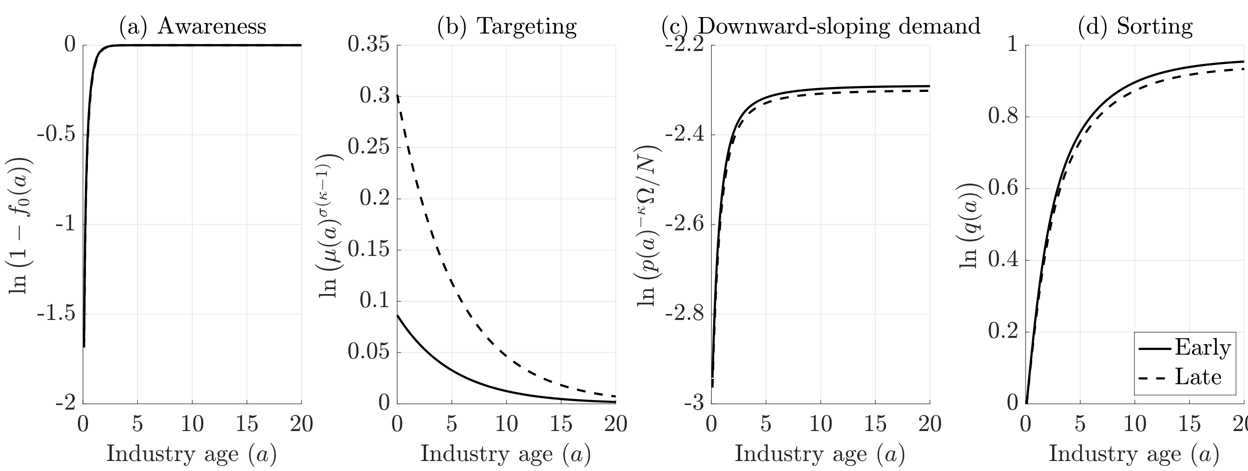


Notes: This figure plots various equilibrium outcomes at the firm level as a function of industry age, for the early calibration (solid line) and the late calibration (dashed line). Panel (a) is targeting, $\mu(a)^{\sigma(\kappa-1)}$, a component in firm-level demand (equation (20)); panel (b) shows prices, $p(a)$, defined in Proposition 3; panel (c) shows total firm-level demand, defined in equation (30); panel (d) shows firm-level profits, defined in equation (32).

calibration early on in an industry's life cycle but, around the fifteenth year of the industry's life, the profits in the early calibration become higher.

To better understand the evolution of firm-level demand over the industry's life cycle, we use the decomposition found in equation (20). When expressed in logs, this equation allows us to write (log) demand as the sum of four components: (i) awareness, i.e., the share of consumers who are aware of at least one product; (ii) targeting; (iii) the intensive-margin downward-sloping component, which captures the price elasticity of intensive demand; and (iv) the extensive-margin sorting component.

Figure 3: Components of firm-level demand (in logs).



Notes: This figure plots the four components of firm demand identified in equation (20), expressed in logs.

We report this log-decomposition for the early and late calibrations in Figure 3. Panel (a) shows that overall product awareness only contributes to demand in the very earliest stages of the industry, and that there is very little difference between the early and late calibrations. Panel (b) shows the difference in targeting between the two economies. As discussed above, the late calibration is characterized by a lower cost of targeting, which results higher demand through more targeting. As can be observed from this panel, targeting plays a major role in explaining the differences in firms' demands between the two calibrations, especially at early stages of the industry. In panel (c) we see that the intensive-margin component, whereby higher income and lower prices result in higher demand, also plays a major role, though the differences between calibrations are small: firms charge higher prices in the late calibration, but income is also higher, and these two forces offset each other. Finally, the sorting component (panel (d)), which relates to the size of the awareness sets of consumers and thus to the extensive margin of demand, exhibits again small differences between calibrations.

In sum, the higher contact rate (θ) in the early calibration implies a higher average number of firms in consumers' awareness sets at any time and leads to higher demand in the early calibration. Early on in the industry's life cycle, the targeting effect dominates and results in higher demand in the late calibration. As targeting converges across calibrations, the intensive and extensive margins of demand (panels (c) and (d) of Figure 3) eventually become the dominant forces, leading to higher demand in the early calibration in the later stages of the industry's life cycle.

4 Counterfactual Experiments

What are the effects of changes in advertising and targeting over time on markups, industry dynamics, match quality, aggregate consumption, and welfare? To investigate this question, we use our calibration results to construct a counterfactual economy for the late period (the year 2014) for which either one, or both, of the advertising cost parameters stay constant relative to the early period (2005). Starting from the late calibration, we re-compute the economy's stationary equilibrium assuming that the cost parameters related to advertising (ν and/or η) are set back to their levels in 2005, but all other parameters remain fixed at their values for the late calibration.

4.1 Effects on Match Quality, Sorting and Markups

Table 3 reports the results from our counterfactual exercises on selected equilibrium variables. Columns (3) and (4) show the results of our first counterfactual experiment, in which we set the values of both ν and η to their values in 2005, and recompute the model's equilibrium leaving all other parameters fixed at their late-period calibrated values. That is, we compute how a counterfactual economy would look like if the advertising technology had not changed at all from 2005 to 2014. Recall that, according to our calibration results, the cost of advertising was higher overall in the early period, coming from both a higher cost of contacting customers ($\nu_{2005} > \nu_{2014}$) as well as a higher cost of targeting ($\eta_{2005} > \eta_{2014}$). This cost difference in both advertising technologies leads firms to choose a lower advertising investment overall in the counterfactual economy: the

Table 3: Counterfactual experiments

	(1) Early (base)	(2) Late (base)	(3) Early ν & η	(4) Change (%)	(5) Early ν only	(6) Change (%)	(7) Early η only	(8) Change (%)
A. Advertising and markups								
Contact rate (θ)	1.924	1.853	1.949	5.16%	1.748	-5.66%	2.066	11.47%
Targeting rate (μ_0)	1.230	2.088	1.236	-40.81%	2.118	1.46%	1.227	-41.22%
Average targeting ($\boldsymbol{\mu}$)	0.023	0.101	0.023	-77.28%	0.108	6.68%	0.021	-79.27%
Average return to targeting	0.048	0.213	0.049	-77.09%	0.227	6.46%	0.045	-78.88%
Average sales-wtd. markup	0.466	0.460	0.457	-0.68%	0.462	0.30%	0.456	-0.93%
Firm value, without adv. cost ($V_t(0)$)	1.057	1.121	1.116	-0.44%	1.126	0.47%	1.111	-0.85%
B. Shares of GDP								
Consumption share ($\boldsymbol{C/Y}$)	0.649	0.647	0.650	0.56%	0.647	0.00%	0.650	0.53%
Advertising share ($\boldsymbol{D/Y}$)	0.022	0.022	0.021	-5.44%	0.023	3.97%	0.020	-8.84%
Industry creation inv. share ($\boldsymbol{I^M/Y}$)	0.185	0.186	0.184	-1.44%	0.186	-0.42%	0.185	-1.00%
Capital investment share ($\boldsymbol{I^K/Y}$)	0.144	0.144	0.145	0.21%	0.144	-0.09%	0.145	0.29%
Profit share ($\boldsymbol{\Pi/Y}$)	0.318	0.315	0.314	-0.46%	0.316	0.21%	0.313	-0.64%
C. Economic aggregates								
Mass of industries ($\boldsymbol{M}$)	1.000	1.000	1.183	18.32%	1.027	2.70%	1.143	14.25%
Wage (w)	2.087	2.205	2.653	20.31%	2.272	3.04%	2.552	15.74%
Consumption level ($\boldsymbol{C}$)	2.977	3.123	3.770	20.72%	3.221	3.14%	3.623	16.02%
Normalized consumption ($\boldsymbol{C/M}$)	2.977	3.123	3.186	2.03%	3.136	0.42%	3.171	1.55%
Match quality ($\boldsymbol{Q}$)	1.474	1.529	1.462	-4.40%	1.519	-0.67%	1.476	-3.51%
Distortion-adjusted quality ($\boldsymbol{QB^{-1}}$)	2.161	2.233	2.130	-4.60%	2.220	-0.58%	2.148	-3.79%
Output level ($\boldsymbol{Y}$)	4.589	4.829	5.798	20.05%	4.981	3.14%	5.574	15.41%
Aggregate TFP ($\boldsymbol{Z}$)	2.161	2.233	2.521	12.88%	2.280	2.11%	2.455	9.92%

Notes: Results from our counterfactual experiments on selected equilibrium variables. Columns (1) and (2) report baseline results for the early (2005) and late (2014) calibrations, respectively (same as Table 2). Column (3) reports 2014 results when both η and ν are fixed at their 2005 values, with column (4) stating the percentage change with respect to the baseline late calibration, i.e., the percentage change of column (3) relative to column (2). Column (5) repeats the experiment but re-setting only the contacting cost parameter ν to its 2005 level, with column (6) stating the percentage change relative to column (2). Column (7) does the same except for the targeting cost parameter η, with column (8) stating the percentage change relative to column (2).

advertising share of GDP is 5.44 percent lower in the counterfactual compared to its baseline 2014 level. However, this decrease hides some heterogeneity. Indeed, investment in targeting is significantly lower (the targeting rate μ_0 is nearly 41 percent lower), but firms choose to contact new customers more frequently (the contact rate θ is 5 percent higher). Since the relative cost of targeting increases (i.e., $\eta_{2005}/\nu_{2005} > \eta_{2014}/\nu_{2014}$), firms substitute more frequent contacts of potentially new customers for lower targeting. In sum, had there been no changes in advertising costs, firms would not have increased their targeting efforts as much as the model predicts they did. Instead, they would contact more consumers, which translates to larger awareness sets and more competition across firms, holding industry age fixed.

The substitution between contacting and targeting is also apparent in columns (5) and (7), where we recompute the 2014 equilibrium but separately reset ν and η back to their 2005 levels, respectively. Raising the cost of contacting new customers, but keeping the cost of targeting fixed (column (5)), leads to a higher level of targeting and a lower contact rate relative to the baseline late-period economy. On the other hand, raising the cost of targeting, but keeping the cost of contacting new customers fixed (column (7)), leads to a lower level of targeting but also to a higher contact rate.

These two counterfactual experiments illustrate some of the opposite effects that changes in the contact rate and in the level of targeting can have on the economy. For instance, a higher cost of contacting new customers increases the level of targeting in the economy, but reduces the contact rate, implying that customers are on average aware of fewer products. This leads to less competition for firms and an increase in the level of markups: the average markup is 0.3 percent higher when ν is returned to its 2005 level. On the other hand, an increase in the cost of targeting leads firms to invest more in contacting customers, raising competition and lowering markups: the average markup is 0.9 percent lower when η is set to its 2005 level. This translates into similar differential effects on the value of firms (before advertising costs) in both experiments. All in all, we find that the average markup would be lower if costs in both dimensions of advertising were at their (higher) level from 2005.

4.2 Effects on Welfare

How did the observed decrease in advertising costs and the rise of digital advertising affect consumer welfare? There are several effects to consider. First, as mentioned above, the changes over time contribute positively to the average markup, due to reduced competition thanks to smaller awareness sets. While this has a negative effect on welfare, the decrease in the cost of advertising (and on targeting, in particular) has overcompensating effects on welfare. For instance, a lower cost of targeting significantly raises the average quality of a consumer-firm match. Column (3) of Table 3 shows that returning to the (higher) level for advertising costs from 2005 would imply a 4.4 percent lower match quality $\boldsymbol{Q}$, and a slightly larger reduction (by 4.6 percent) in distortion-adjusted quality $\boldsymbol{QB}^{-1}$ due to increased markup dispersion. Finally, there is also the general equilibrium effects to consider. If the advertising technology had remained unchanged, the rate at which new industries are created would be higher, implying a higher steady-state mass of industries, which translates into a higher level of output and consumption.

Taking all of these forces into account, we find that bringing back the cost of advertising to its level in 2005 would raise consumption-equivalent welfare considerably at 20.7 percent, due to a large degree to an expansion in the measure of available product categories, $\boldsymbol{M}$, which increases by 18.3 percent.[29] Even if we ignore this effect that is due to taste for variety, and focus on the normalized consumption $\boldsymbol{C}/\boldsymbol{M}$, the gain in consumption-equivalent welfare is still quite significant at 2.03 percent. That is, even though the average consumer-firm match quality is higher in the

[29] These large welfare effects owe primarily to the linear industry creation technology, which is very elastic to changes in industry value. The use of a convex cost function would diminish the calculated welfare changes considerably, so the exact magnitudes should be interpreted as upper bounds.

actual 2014 economy, consumers would still be better off if the advertising technology had remained unchanged.

It should also be noted that the negative welfare effect owes much more to the lower cost of targeting than to that of contacting. Most of the change in the normalized level of consumption is attributable to the effect of the change in η in column (7) of Table 3 (1.55 percent), rather than that of ν in column (5) (0.42 percent). This means the rise in digital advertising, and the better targeting technology it provides, is the main culprit.

Summing up, while a lower cost of advertising from the early to the late period allowed for better matches between consumer preferences and the products they are exposed to through advertising, it was also associated with an increase in the average level of markups and a decrease in new industry creation that offsets the beneficial effect of better targeting on welfare.

5 Conclusion

The rise of digital technologies over the last few decades is shaping the way firms advertise their products. Technological advances in the advertising sector have not only affected the rate at which firms can contact new consumers via spreading product awareness, but also how firms' products are tailored to consumers' preferences via targeted advertising.

In this paper, we develop a general-equilibrium, heterogeneous-agent model of demand as a network, in which consumers become slowly aware of products. In this framework, advertising can affect both the speed at which buyer-seller networks are formed as well as the quality of the matches, i.e., how strongly a firm's product is correlated with the customer's idiosyncratic preference for it. While faster network formation leads to stronger competition and lower markups over the industry's life cycle through larger awareness sets (better sorting), higher targeting allows firms to efficiently segment consumers and extract surplus while maintaining high prices. In this framework, the advent of digital advertising, though expected to have a stronger impact on targeting, may therefore have ambiguous implications for competition, industry dynamics, and welfare.

To quantitatively assess the net effects of the rise in digital advertising, we discipline the model by calibrating it to two different time periods, 2005 and 2014. Over this time, the share of digital advertising spending in total advertising rose considerably, from 6 to 31 percent. In the estimated model, this rise in digital advertising is associated with a decrease in both the cost of contacting and the cost of targeting, and it implies an increase in aggregate TFP and welfare through a rise in match quality, in spite of slightly worse consumer sorting. However, counterfactual experiments reveal that if the advertising technology had not improved during this period, aggregate distortions due to market power would have been lower, partially offsetting the benefits that better targeting had on welfare through improved match quality.

Our analysis suggests that policy-makers should consider the role of digital advertising on market power in addition to privacy concerns —especially considering that social network and mobile advertising, where targeting is especially prevalent, is expected to become the dominant form of advertising in the next few years. Even though digital advertising may enable consumers to find products whose characteristics are more aligned with their preferences, it could also lead to significant increases in market power via market segmentation and worse consumer sorting. These changes can also negatively impact the creation of new product categories and business dynamism, as demonstrated in our counterfactual experiments. In light of these findings, our study suggests that the appropriate regulations may need to offer incentives for efficient targeted advertising without limiting consumer awareness excessively. Exploring these and other policy questions remains an exciting avenue for future work.

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A Model of Product Awareness and Industry Life Cycles

by Laurent Cavenaile, Murat Alp Celik, Jesse Perla and Pau Roldan-Blanco

Online Appendix

A Proofs

A.1 Proof of Proposition 1

Proof. The static problem of the consumer is to allocate expenditures y_{imjt} to solve:

$$\max_{(y_{imjt} \geq 0)} \underbrace{\left[\int_0^{\boldsymbol{M}_t} \left(\sum_{i \in A_{mjt}} \overline{\Gamma} e^{\sigma \xi_{imj}} y_{imjt} \right)^{\frac{\kappa-1}{\kappa}} \mathrm{d}m \right]^{\frac{\kappa}{\kappa-1}}}_{\equiv Y_{jt}} \quad \text{s.t.} \int_0^{\boldsymbol{M}_t} \sum_{i \in A_{mjt}} \widehat{p}_{imt} y_{imjt} \mathrm{d}m \leq P_{jt} \Omega_{jt}. \tag{A.1.1}$$

The Lagrangian of this problem is:

$$\begin{aligned} \mathscr{L}_{jt} = & \left[\int_0^{\boldsymbol{M}_t} \left(\sum_{i \in A_{mjt}} \overline{\Gamma} e^{\sigma \xi_{imj}} y_{imjt} \right)^{\frac{\kappa-1}{\kappa}} \mathrm{d}m \right]^{\frac{\kappa}{\kappa-1}} - \lambda_{jt} \left(\int_0^{\boldsymbol{M}_t} \sum_{i \in A_{mjt}} \widehat{p}_{imt} y_{imjt} \mathrm{d}m - P_{jt} \Omega_{jt} \right) \\ & + \int_0^{\boldsymbol{M}_t} \sum_{i \in A_{mjt}} \vartheta_{imjt} y_{imjt} \mathrm{d}m, \end{aligned} \tag{A.1.2}$$

where $\lambda_{jt} \geq 0$ is the Lagrange multiplier on the budget constraint, and $\left(\{ \vartheta_{imjt} \geq 0 \}_{i \in A_{mj}} : m \in [0, \boldsymbol{M}_t] \right)$ are the multipliers ensuring weak positivity on every choice of y_{imjt}. The first-order condition is:

$$\left(\frac{Y_{jt}}{\sum_{i \in A_{mjt}} \overline{\Gamma} e^{\sigma \xi_{imj}} y_{imjt}} \right)^{\frac{1}{\kappa}} \overline{\Gamma} e^{\sigma \xi_{imj}} = \lambda_{jt} \widehat{p}_{imt} - \vartheta_{imjt}. \tag{A.1.3}$$

By monotonicity of preferences and the complementary slackness condition, we have $\lambda_{jt} > 0$, and $\vartheta_{imjt} y_{imjt} = 0$, with $\vartheta_{imjt} \geq 0$, $\forall (i, m)$. We conjecture (to be verified later) that if $m \in \mathcal{M}_{jt} \equiv \{ m : A_{mjt} \neq \emptyset \} \subseteq [0, \boldsymbol{M}_t]$, the consumer will almost surely purchase at most one product from industry m. Denote this product by $i(m) \in A_{mjt}$. Then, from equation (A.1.3), we have:

$$\forall m \in \mathcal{M}_{jt} : \quad Y_{jt}^{\frac{1}{\kappa}} \left(\overline{\Gamma} e^{\sigma \xi_{i(m)mj}} \right)^{\frac{\kappa-1}{\kappa}} y_{i(m)mjt}^{-\frac{1}{\kappa}} = \lambda_{jt} \widehat{p}_{i(m)mt}. \tag{A.1.4}$$

Taking the ratio between any two industries $m, m' \in \mathcal{M}_{jt}$ yields the relative demand function:

$$y_{i(m')m'jt} = y_{i(m)mjt} e^{\sigma(\kappa-1)\left(\xi_{i(m')m'j} - \xi_{i(m)mj}\right)} \left(\frac{\widehat{p}_{i(m)mt}}{\widehat{p}_{i(m')m't}}\right)^{\kappa}. \tag{A.1.5}$$

Multiplying both sides by $\widehat{p}_{i(m')m't}$ and integrating over all industries with positive purchased quantities (i.e., in the $\mathcal{M}_{jt}$ set), we obtain total expenditures for consumer j at time t:

$$P_{jt}\Omega_{jt} = \int_{\mathcal{M}_{jt}} \widehat{p}_{i(m')m't} y_{i(m')m'jt} \mathrm{d}m' \tag{A.1.6}$$

$$= y_{i(m)mjt} \widehat{p}^{\kappa}_{i(m)mt} e^{\sigma\xi_{i(m)mj}(1-\kappa)} \int_{\mathcal{M}_{jt}} \left(e^{-\sigma\xi_{i(m')m'j}} \widehat{p}_{i(m')m't}\right)^{1-\kappa} \mathrm{d}m', \tag{A.1.7}$$

where the first equality comes from the fact that $\lambda_{jt} > 0$ (so that the budget constraint is always binding), and on the right-hand side we have used that industry m is infinitesimal to pull functions of m out of the integral. Next, define the price index P_{jt} as in equation (16), that is:

$$P_{jt} \equiv \overline{\Gamma}^{-1} \left(\int_{\mathcal{M}_{jt}} \left(e^{-\sigma\xi_{i(m')m'j}} \widehat{p}_{i(m')m't}\right)^{1-\kappa} \mathrm{d}m'\right)^{\frac{1}{1-\kappa}}. \tag{A.1.8}$$

This allows us to write equation (A.1.7) as:

$$P_{jt}\Omega_{jt} = y_{i(m)mjt} \widehat{p}^{\kappa}_{i(m)mt} \overline{\Gamma}^{1-\kappa} e^{-\sigma(\kappa-1)\xi_{i(m)mj}} P_{jt}^{-(\kappa-1)}. \tag{A.1.9}$$

Rearranging, and defining real prices as $p_{i(m)mjt} \equiv \widehat{p}_{i(m)mt}/P_{jt}$, we find the intensive demand function for product i in industry m:

$$y^{d}_{i(m)mjt} = \overline{\Gamma}^{\kappa-1} e^{\sigma(\kappa-1)\xi_{i(m)mj}} p^{-\kappa}_{i(m)mjt} \Omega_{jt}. \tag{A.1.10}$$

This shows part 2 of Proposition 1. To show part 1, and thereby confirm our initial conjecture that the individual consumes at most one product from each industry, fix an industry m and take two products, i and $i' \in \mathcal{M}_{jt} \backslash \{i\}$. From our initial conjecture, $y_{i'mjt} = 0$. Then, from (A.1.3), we have that:

$$Y_{jt}^{\frac{1}{\kappa}} \left(\overline{\Gamma} e^{\sigma\xi_{imj}} y_{imjt}\right)^{-\frac{1}{\kappa}} \overline{\Gamma} e^{\sigma\xi_{i'mj}} \leq \lambda_{jt} \widehat{p}_{i'mt}, \tag{A.1.11}$$

where, rearranging from equation (A.1.4), we know:

$$Y_{jt}^{\frac{1}{\kappa}} \left(\overline{\Gamma} e^{\sigma\xi_{imj}} y_{imjt}\right)^{-\frac{1}{\kappa}} = \lambda_{jt} \frac{\widehat{p}_{imt}}{\overline{\Gamma} e^{\sigma\xi_{imj}}}. \tag{A.1.12}$$

Back into equation (A.1.11) and taking logs, we obtain:

$$\ln\left(\frac{\widehat{p}_{i'mt}}{\widehat{p}_{imt}}\right) \geq \sigma(\xi_{i'mj} - \xi_{imj}). \tag{A.1.13}$$

Using $p_{imjt} \equiv \widehat{p}_{imt}/P_{jt}$, equation (A.1.13) gives equation (14) in Proposition 1. To finish the proof, we need to confirm our initial conjecture that the individual purchases only one product, almost surely, from each industry in which the consumer is aware of at least one firm. To confirm the conjecture, we show that the set of consumers that choose two or more products per industry is measure zero. Suppose, instead, that there is a non-empty subset $\mathcal{J} \subseteq [0,1]$ such that, for all $j \in \mathcal{J}$, we can find some industry $m \in \mathcal{M}_{jt}$ where $y_{i_1,mjt}, \ldots, y_{i_k,mjt} > 0$ for some subset $\{i_1, \ldots, i_k\} \subseteq A_{mjt}$ with $2 \leq k \leq N$. Then, the optimality conditions for each $n = 1, \ldots, k$ are:

$$\lambda_{jt}\widehat{p}_{i_1,mt} = Y_{jt}^{\frac{1}{\kappa}} \left(\sum_{n=1}^{k} \overline{\Gamma} e^{\sigma \xi_{i_n,mj}} y_{i_n,mjt} \right)^{-\frac{1}{\kappa}} \overline{\Gamma} e^{\sigma \xi_{i_1,mj}}, \tag{A.1.14}$$

$$\lambda_{jt}\widehat{p}_{i_2,mt} = Y_{jt}^{\frac{1}{\kappa}} \left(\sum_{n=1}^{k} \overline{\Gamma} e^{\sigma \xi_{i_n,mj}} y_{i_n,mjt} \right)^{-\frac{1}{\kappa}} \overline{\Gamma} e^{\sigma \xi_{i_2,mj}}, \tag{A.1.15}$$

$$\vdots$$

$$\lambda_{jt}\widehat{p}_{i_k,mt} = Y_{jt}^{\frac{1}{\kappa}} \left(\sum_{n=1}^{k} \overline{\Gamma} e^{\sigma \xi_{i_n,mj}} y_{i_n,mjt} \right)^{-\frac{1}{\kappa}} \overline{\Gamma} e^{\sigma \xi_{i_k,mj}}. \tag{A.1.16}$$

Taking the ratio of any two $r, q \in \{2, \ldots, k\}$ yields:

$$\frac{\widehat{p}_{i_r,mt}}{\widehat{p}_{i_q,mt}} = e^{\sigma(\xi_{i_r,mj} - \xi_{i_q,mj})}. \tag{A.1.17}$$

Taking logs:

$$\ln\left(\frac{\widehat{p}_{i_r,mt}}{\widehat{p}_{i_q,mt}}\right) = \sigma(\xi_{i_r,mj} - \xi_{i_q,mj}). \tag{A.1.18}$$

For a given set of positive prices and for a given distribution of ξ, there is a positive measure of individuals with this particular combination of $(\xi_{i_r,mj}, \xi_{i_q,mj})$. However, the solution is an affine subset of the $\{\xi_{imj}\}$ space, and given the independence of the ξ preferences (by Assumption 1), this affine subset must have measure zero, a contradiction with our initial assertion. Therefore, we conclude that the set of agents who purchase multiple products is empty if prices are positive, and thus equation (A.1.13) can be written as a strict inequality for almost every consumer.

□

A.2 Proof of Proposition 2

Proof. According to the definition in equation (17), the firm must integrate across those consumers whose awareness sets include the firm and who, in addition, choose the firm's product over all other products that they are aware of. This requires integrating over all awareness sets $A \in \mathcal{A}_i(a)$ (where $\mathcal{A}_i(a)$ was defined in equation (6)), as well as over idiosyncratic preferences not only for the firm's product, but also for all the other products in each awareness set. Denote by $n \equiv |A| \in \mathcal{I}$ the size of a typical awareness set, and let $\Psi_i(a, A, \vec{\xi}(A))$ be the cumulative density function (cdf) faced by firm i at industry age a that corresponds to the joint distribution of (i) awareness sets that contain the firm, $A \in \mathcal{A}_i(a)$, and (ii) preference shifters across all products in the awareness set, $\vec{\xi}(A) \equiv [\xi_1, \ldots, \xi_i, \ldots, \xi_n]^\top \in \mathbb{R}^n$. Using Bayes' theorem, we can factor this joint density into the marginal density of awareness sets that contain the firm, denoted $\widehat{f}(a, A)$, and a conditional density of preferences for each given awareness set, denoted $\mathrm{d}H_i(a, \vec{\xi}(A)|A)$.[30] That is:

$$\mathrm{d}\Psi_i(a, A, \vec{\xi}(A)) = \widehat{f}(a, A)\mathrm{d}H_i(a, \vec{\xi}(A)|A). \tag{A.2.1}$$

Using our assumptions, these expressions can be simplified further. First, by Assumption 2, network connections form independently of existing connections. Moreover, in a symmetric equilibrium all firms have the same arrival rate of forming (θ) and losing (ζ) connections. Therefore, for all sets $A, A' \subseteq \mathcal{I}$ with $|A| = |A'|$, we have $\widehat{f}(a, A) = \widehat{f}(a, A')$, for all a. This means, in particular, that the probability that firm i is in an awareness set A of size n, $\widehat{f}(a, A)$, can be described as an urn-like problem *without* replacement, as the same firm cannot be drawn again after it has been first introduced into the awareness set. This probability is described by the Hypergeometric distribution. In particular, the probability of a success event (i.e., firm $i = 1, \ldots, N$ is drawn once and without replacement into a subset of firms of size $n \leq N$) equals $\frac{\binom{n}{1}\binom{N-n}{0}}{\binom{N}{1}} = \frac{n}{N}$. Recall that the proportion of consumers aware of n products in an industry of age a was defined as $f_n(a)$ in the main text, and follows the law of motion given in equation (4). Therefore, we can write:

$$\widehat{f}(a, A) = \frac{n}{N} f_n(a). \tag{A.2.2}$$

This result shows that, the probability that a firm can be found in a given awareness set A is only a function of the *size* of the set, $n = |A|$, but not on the *composition* of this set.

Second, by Assumption 1, idiosyncratic preferences are independently and identically distributed and, in particular, unrelated to the evolution of awareness. Hence, we can write the conditional density $\mathrm{d}H_i(a, \vec{\xi}(A)|A)$ as the product of marginal densities of preference shifters, for each product $i \in A$. As idiosyncratic preferences are Gumbel-distributed with firm-specific mean $\widetilde{\mu}_i$, we have:

$$\mathrm{d}H_i(a, \vec{\xi}(A)|A) = \mathrm{d}G(\xi_i; \widetilde{\mu}_i) \prod_{i' \in A \setminus \{i\}} \mathrm{d}G(\xi_{i'}; \widetilde{\mu}_{i'}), \tag{A.2.3}$$

where $G(\cdot; \widetilde{\mu}_i)$ denotes the cdf of the Gumbel distribution with location parameter $\widetilde{\mu}_i$ and scale parameter equal to one. For analytical convenience, let us re-center this distribution by defining $\mu_i \equiv e^{\widetilde{\mu}_i}, \forall i \in \mathcal{I}$. Henceforth, we will write the firm's problem in terms of μ's instead of $\widetilde{\mu}$'s. Using

[30] Note that we do not index $\widehat{f}$ by the firm's identity i because all firm dependence is already encoded in the awareness set A, which the firm belongs to. However, H_i does depend explicitly on i as, conditional on the awareness set, the consumer has heterogeneous preferences over each firm.

the formula for the cdf of a Gumbel distribution, we can write:

$$G(\xi;\mu) = e^{-e^{-\xi+\ln\mu}} = e^{-\mu e^{-\xi}}, \tag{A.2.4a}$$

$$G(\xi;\mu_{i'}) = e^{-e^{-\xi+\ln\mu_{i'}}} = e^{-e^{-\xi}} e^{-(\mu_{i'}-1)e^{-\xi}}, \quad \forall i' \in A\backslash\{i\}, \tag{A.2.4b}$$

where μ is firm i's targeting. To make progress toward writing out the total demand of the firm, let us conjecture (a claim that we will verify in Proposition 5) that both real income and the price index are identical across consumers, $\Omega_{jt} = \boldsymbol{\Omega}_t$ and $P_{jt} = \boldsymbol{P}_t$. This implies that real prices are constant across consumers as well. Then, using equations (15) and (17), and taking as given the vectors of *real* prices $\vec{p}_{-i} \equiv \{p_{i'}\}_{i'\neq i}$ and match qualities $\vec{\mu}_{-i} \equiv \{\mu_{i'}\}_{i'\neq i}$ from competitors, we can write the demand in industry m of age a faced by firm i choosing real price p and a level of targeting μ as:[31]

$$y_{imt}(a,p,\mu;\vec{p}_{-i},\vec{\mu}_{-i}) = \overline{\Gamma}^{\kappa-1} p^{-\kappa} \boldsymbol{\Omega}_t \sum_{A\in\mathcal{A}_{im}(a)} \widehat{f}(a,A)\phi_i(a,A,p,\mu;\vec{p}_{-i},\vec{\mu}_{-i}), \tag{A.2.5}$$

where $\phi_i(a,A,p,\mu;\vec{p}_{-i},\vec{\mu}_{-i})$ is short-hand notation for the total demand, up to the constant $\overline{\Gamma}^{\kappa-1} p^{-\kappa}\boldsymbol{\Omega}_t$, from consumers with awareness set A, defined by:

$$\phi_i(a,A,p;\vec{p}_{-i},\vec{\mu}_{-i}) \equiv \int_{\mathbb{R}^n} e^{\sigma(\kappa-1)\xi_i} \mathbb{1}\left\{\ln\left(\frac{p_{i'}}{p}\right) > \sigma(\xi_{i'}-\xi_i) \middle| \forall i' \in A\backslash\{i\}\right\} \mathrm{d}H_i(a,\vec{\xi}(A)|A), \tag{A.2.6}$$

where by the symbol $\int_{\mathbb{R}^n}$ we mean $\int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty}$, with $n = |A|$ such successive integrals. Let us write $\phi_i \equiv \phi_i(a,A,p,\mu;\vec{p}_{-i},\vec{\mu}_{-i})$ to shorten notation. Using independence (equation (A.2.3)), we can write (A.2.6) as follows:

$$\phi_i = \int_{\mathbb{R}^n} e^{\sigma(\kappa-1)\xi_i} \mathbb{1}\left\{\frac{1}{\sigma}\ln\left(\frac{p_{i'}}{p}\right) + \xi_i > \xi_{i'} \middle| \forall i' \in A\backslash\{i\}\right\} \mathrm{d}G(\xi_i;\mu) \prod_{i'\in A\backslash\{i\}} \mathrm{d}G(\xi_{i'};\mu_{i'}). \tag{A.2.7}$$

Using Fubini's Theorem, we can separate out the n-tuple integral into two iterated integrals:

$$\phi_i = \int_{-\infty}^{+\infty} e^{\sigma(\kappa-1)\xi_i} \left[\int_{\mathbb{R}^{n-1}} \mathbb{1}\left\{\frac{1}{\sigma}\ln\left(\frac{p_{i'}}{p}\right) + \xi_i > \xi_{i'} \middle| \forall i' \in A\backslash\{i\}\right\} \prod_{i'\in A\backslash\{i\}} \mathrm{d}G(\xi_{i'};\mu_{i'})\right] \mathrm{d}G(\xi_i;\mu). \tag{A.2.8}$$

Notice that, inside the square brackets, we must compute the cdf of the joint distribution of $\vec{\xi}\backslash\{\xi_i\}$ at the point $\frac{1}{\sigma}\ln\left(\frac{p_{i'}}{p}\right) + \xi_i$. Thus, using the pdf's associated to (A.2.4a)-(A.2.4b), we have:

[31] For the derivation to follow, it is convenient to write $\widehat{f}(a,A)$ without imposing equation (A.2.2) yet. We will use result (A.2.2) as the final step of the proof.

$$\phi_i = \int_{-\infty}^{+\infty} e^{\sigma(\kappa-1)\xi_i} \left[\prod_{i' \in A \setminus \{i\}} e^{-e^{-\left(\frac{1}{\sigma}\ln\left(\frac{p_{i'}}{p}\right)+\xi_i\right)+\ln(\mu_{i'})}} \right] e^{-\xi_i} e^{-e^{-\xi_i}} \mu e^{-(\mu-1)e^{-\xi_i}} \mathrm{d}\xi_i$$

$$= \int_{-\infty}^{+\infty} e^{\sigma(\kappa-1)\xi_i} e^{-\sum_{i' \in A \setminus \{i\}} e^{-\left(\frac{1}{\sigma}\ln\left(\frac{p_{i'}}{p}\right)+\xi_i\right)+\ln(\mu_{i'})}} e^{-\xi_i} e^{-e^{-\xi_i}} \mu e^{-(\mu-1)e^{-\xi_i}} \mathrm{d}\xi_i. \tag{A.2.9}$$

Next, notice that we can factor the term $e^{-e^{-\xi_i}}$ in (A.2.9) as follows:

$$e^{-e^{-\xi_i}} = e^{-e^{-\left(\frac{1}{\sigma}\ln\left(\frac{p}{p}\right)+\xi_i\right)}} = e^{-e^{-\left(\frac{1}{\sigma}\ln\left(\frac{p}{p}\right)+\xi_i\right)+\ln(\mu_{i'})}} e^{(\mu_{i'}-1)e^{-\xi_i}}. \tag{A.2.10}$$

This is useful because it allows us to have that the product term in (A.2.9) is taken over all the products in A, including i. In particular, by Assumption 4 we have $\mu_{i'} = \mu_{-i}$, $\forall i' \in A \setminus \{i\}$, which allows us to write:

$$\phi_i = \int_{-\infty}^{+\infty} e^{\sigma(\kappa-1)\xi_i} e^{-\sum_{i' \in A} e^{-\left(\frac{1}{\sigma}\ln\left(\frac{p_{i'}}{p}\right)+\xi_i\right)+\ln(\mu_{-i})}} e^{-\xi_i} \mu e^{-(\mu-\mu_{-i})e^{-\xi_i}} \mathrm{d}\xi_i. \tag{A.2.11}$$

Simplifying by factoring exponential terms together:

$$\phi_i = \int_{-\infty}^{+\infty} \mu e^{-\left(1-\sigma(\kappa-1)\right)\xi_i} e^{-e^{-\xi_i}\left(\mu-\mu_{-i}+\mu_{-i}\sum_{i' \in A}\left(\frac{p_{i'}}{p}\right)^{-\frac{1}{\sigma}}\right)} \mathrm{d}\xi_i. \tag{A.2.12}$$

Next, we can use the fact that $\int_{-\infty}^{+\infty} e^{-a_1 x} e^{-a_2 e^{-x}} \mathrm{d}x = a_2^{-a_1}\Gamma(a_1)$, for any two numbers $a_1, a_2 > 0$, where $\Gamma(.)$ is the Gamma function. Using this logic into the last equation then gives us our final result for the demand from consumers with awareness set A:

$$\phi_i(a, A, p, \mu; \vec{p}_{-i}, \vec{\mu}_{-i}) = \overline{\Gamma}^{1-\kappa} \mu^{\sigma(\kappa-1)} \left(1 - \frac{\mu_{-i}}{\mu} + \frac{\mu_{-i}}{\mu} \sum_{i' \in A} \left(\frac{p_{i'}}{p}\right)^{-\frac{1}{\sigma}}\right)^{\sigma(\kappa-1)-1}, \tag{A.2.13}$$

where $\overline{\Gamma} \equiv \Gamma\big(1-\sigma(\kappa-1)\big)^{\frac{1}{1-\kappa}}$. Finally, back into (A.2.5), we obtain the formula:

$$y_{imt}(a, p, \mu; \vec{p}_{-i}, \vec{\mu}_{-i}) = \mu^{\sigma(\kappa-1)} p^{-\kappa} \boldsymbol{\Omega}_t \sum_{A \in \mathcal{A}_{im}(a)} \widehat{f}(a, A) \left(1 + \frac{\mu_{-i}}{\mu} \sum_{i' \in A \setminus \{i\}} \left(\frac{p_{i'}}{p}\right)^{-\frac{1}{\sigma}}\right)^{\sigma(\kappa-1)-1}. \tag{A.2.14}$$

As a final step, we can replace $\widehat{f}(a, A)$ by $\frac{n}{N} f_n(a)$ using result (A.2.2). Inside of (A.2.14), this allows us to have a sum over all possible awareness set sizes rather than over the sets themselves. With symmetry in prices, $p_{i'} = p_{-i}$ for all $i' \neq i$ and some $p_{-i} > 0$, we obtain:

$$y_{it}(a,p,\mu;\vec{p}_{-i},\vec{\mu}_{-i}) = \mu^{\sigma(\kappa-1)} p^{-\kappa} \frac{\mathbf{\Omega}_t}{N} \sum_{n=1}^{N} n f_n(a) \left(1 + (n-1)\frac{\mu_{-i}}{\mu}\left(\frac{p_{-i}}{p}\right)^{-\frac{1}{\sigma}}\right)^{\sigma(\kappa-1)-1}. \quad \text{(A.2.15)}$$

To arrive at the expression in Proposition 2, rewrite the last equation as

$$y_{it}(a,p,\mu;\vec{p}_{-i},\vec{\mu}_{-i}) = \mu^{\sigma(\kappa-1)} p^{-\kappa} \frac{\mathbf{\Omega}_t}{N} \sum_{n=1}^{N} f_n(a) g(n), \quad \text{(A.2.16)}$$

where $g(n) \equiv n \left(1 + (n-1)\frac{\mu_{-i}}{\mu}\left(\frac{p_{-i}}{p}\right)^{-\frac{1}{\sigma}}\right)^{\sigma(\kappa-1)-1}$, and notice

$$\sum_{n=1}^{N} f_n(a) g(n) = (1 - f_0(a))\mathbb{E}_a[g(\widehat{n})], \quad \text{(A.2.17)}$$

where $\mathbb{E}_a[g(\widehat{n})]$ is the expectation as of time a of $g(\widehat{n})$, with $\widehat{n} \equiv n | n \geq 1$ (recall equation (18)). Therefore:

$$y_{it}(a,p,\mu;\vec{p}_{-i},\vec{\mu}_{-i}) = (1 - f_0(a))\mu^{\sigma(\kappa-1)} p^{-\kappa} \frac{\mathbf{\Omega}_t}{N} \mathbb{E}_a\left[\widehat{n}\left(1 + (\widehat{n}-1)\frac{\mu_{-i}}{\mu}\left(\frac{p_{-i}}{p}\right)^{-\frac{1}{\sigma}}\right)^{\sigma(\kappa-1)-1}\right], \quad \text{(A.2.18)}$$

our desired result.

□

A.3 Proof of Proposition 3

Proof. Assuming symmetric prices among competitors and using the firm's demand function from equation (19), the problem is:

$$p_i(a) = \arg\max_{p \geq 0} \left\{ \left(p - \boldsymbol{mc}_t\right)(1 - f_0(a))\mu(a)^{\sigma(\kappa-1)} p^{-\kappa} \frac{\mathbf{\Omega}_t}{N} q(a,p;p_{-i},\mu_{-i}) \right\}, \quad \text{(A.3.1)}$$

where $\mu(a) = \mu_0 e^{1-s(a)}$ is the firm's chosen level of targeting, and recall that:

$$q(a,p;p_{-i},\mu_{-i}) \equiv \mathbb{E}_a\left[\widehat{n}\left(1 + (\widehat{n}-1)\frac{\mu_{-i}}{\mu(a)}\left(\frac{p_{-i}}{p_i}\right)^{-\frac{1}{\sigma}}\right)^{\sigma(\kappa-1)-1}\right] \quad \text{(A.3.2)}$$

is the sorting component of demand. Assuming existence of a solution, the first-order condition of (A.3.1) with respect to p can be written as follows:

$$1 + \left(1 - \Lambda(a)^{-1}\right)\left[\mathcal{E}(a,p;p_{-i},\mu_{-i}) - \kappa\right] = 0, \quad \text{(A.3.3)}$$

where we have defined $\Lambda(a) \equiv p/\boldsymbol{mc}_t$ as the markup, and

$$\mathcal{E}(a,p;p_{-i},\mu_{-i}) \equiv p\frac{\partial_p q(a,p;p_{-i},\mu_{-i})}{q(a,p;p_{-i},\mu_{-i})} \tag{A.3.4}$$

as the price-elasticity of the sorting component of demand. Differentiating equation (A.3.2):

$$\mathcal{E}(a,p;p_{-i},\mu_{-i}) = -\frac{1-\sigma(\kappa-1)}{\sigma}\frac{\mathbb{E}_a\left[\widehat{n}(\widehat{n}-1)\frac{\mu_{-i}}{\mu(a)}\left(\frac{p_{-i}}{p}\right)^{-\frac{1}{\sigma}}\left(1+(\widehat{n}-1)\frac{\mu_{-i}}{\mu(a)}\left(\frac{p_{-i}}{p}\right)^{-\frac{1}{\sigma}}\right)^{\sigma(\kappa-1)-2}\right]}{\mathbb{E}_a\left[\widehat{n}\left(1+(\widehat{n}-1)\frac{\mu_{-i}}{\mu(a)}\left(\frac{p_{-i}}{p_i}\right)^{-\frac{1}{\sigma}}\right)^{\sigma(\kappa-1)-1}\right]}. \tag{A.3.5}$$

Specializing this to a symmetric equilibrium with $\mu_{-i} = \mu(a)$ and $p = p_{-i}$, we obtain:

$$\mathcal{E}(a) = -\frac{1-\sigma(\kappa-1)}{\sigma}\left[1-\frac{\mathbb{E}_a\left[\widehat{n}^{\sigma(\kappa-1)-1}\right]}{\mathbb{E}_a\left[\widehat{n}^{\sigma(\kappa-1)}\right]}\right], \tag{A.3.6}$$

Finally, solving for $\Lambda(a)$ in equation (A.3.3) gives our desired result:

$$\Lambda(a) = 1+\frac{1}{\kappa-1+\mathcal{E}(a)}, \tag{A.3.7}$$

□

A.4 Proof of Proposition 4

Proof. The first-order conditions of problem (23) are:

$$r_t = \alpha\boldsymbol{mc}_t\frac{y(a)}{k(a)} - \delta_K, \tag{A.4.1}$$

$$w_t = (1-\alpha)\boldsymbol{mc}_t\frac{y(a)}{l(a)}, \tag{A.4.2}$$

where $\boldsymbol{mc}_t > 0$ is the Lagrange multiplier, equal to the marginal cost.[32] To find the value of this multiplier, use the production function to write capital as:

$$k(a) = \left(\frac{y}{zl(a)^{1-\alpha}}\right)^{\frac{1}{\alpha}}, \tag{A.4.3}$$

and re-write the problem as a choice over labor only:

[32]To show that the marginal cost coincides with the Lagrange multiplier, substitute (A.4.1)-(A.4.2) back into the objective function of problem (23) to find that $\boldsymbol{TC}_t(y,a) = \boldsymbol{mc}_t y$, and therefore $\boldsymbol{mc}_t = \partial_y \boldsymbol{TC}_t(y,a)$.

$$\min_{l(a)} \left\{ (r_t + \delta_K) \left(\frac{y}{zl(a)^{1-\alpha}} \right)^{\frac{1}{\alpha}} + w_t l(a) \right\}. \tag{A.4.4}$$

Taking the first-order condition of sub-problem (A.4.4) we find:

$$l(a) = \left(\frac{1-\alpha}{\alpha} \frac{r_t + \delta_K}{w_t} \right)^{\alpha} \frac{y}{z}, \tag{A.4.5}$$

and using this inside (A.4.3) gives:

$$k(a) = \left(\frac{1-\alpha}{\alpha} \frac{r_t + \delta_K}{w_t} \right)^{\alpha-1} \frac{y}{z}. \tag{A.4.6}$$

Substituting these last two results into the objective function, we find:

$$\boldsymbol{TC}_t(y,a) = (r_t + \delta_K)k(a) + w_t l(a) = \left(\frac{r_t + \delta_K}{\alpha} \right)^{\alpha} \left(\frac{w_t}{1-\alpha} \right)^{1-\alpha} \frac{y}{z}. \tag{A.4.7}$$

Taking the derivative with respect to y, we readily obtain equation (24) in the main text. To find the demands for labor and capital, take the ratio of (A.4.6) and (A.4.5) to find the optimal capital-labor ratio:

$$\widetilde{k}_t \equiv \frac{k(a)}{l(a)} = \frac{\alpha}{1-\alpha} \frac{w_t}{r_t + \delta_K}. \tag{A.4.8}$$

Notice that the capital-labor ratio is constant across industries. Using equation (24):

$$\boldsymbol{mc}_t = \frac{1}{1-\alpha} \frac{w_t}{z} \widetilde{k}_t^{-\alpha}. \tag{A.4.9}$$

Thus, the optimal labor input choice is $l(a,y) = \frac{y}{z}\widetilde{k}_t^{-\alpha}$ or, using equation (A.4.9):

$$l(a,y) = (1-\alpha)\boldsymbol{mc}_t \frac{y}{w_t}. \tag{A.4.10}$$

The optimal capital input choice is then:

$$k(a,y) = \widetilde{k}_t l(a,y). \tag{A.4.11}$$

Finally, in a symmetric equilibrium, from Proposition 2 we know that:

$$y(a) = (1 - f_0(a))\mu(a)^{\sigma(\kappa-1)} p(a)^{-\kappa} \frac{\boldsymbol{\Omega}_t}{N} \mathbb{E}_a \left[\widehat{n} \left(1 + (\widehat{n}-1) \frac{\mu_{-i}}{\mu(a)} \right)^{\sigma(\kappa-1)-1} \right], \tag{A.4.12}$$

with $p(a) = \Lambda(a)\boldsymbol{mc}_t$ from Proposition 3, and $\Lambda(a)$ given by equation (26). The optimal labor and capital demands are then $l(a) = l(a, y(a))$ and $k(a) = \widetilde{k}_t l(a, y(a))$, with which we obtain our desired results.

□

A.5 Proof of Proposition 5

Proof. To show part 1, recall that the price index of consumer j at time t is defined by equation (16). Denote the age of a given industry m by $a(m)$. Given $\boldsymbol{mc}_t$, the equilibrium price of the single product $i(m)$ that the consumer purchases in industry m is only a function of industry age by Proposition 3. Therefore, we can write:

$$P_{jt} = \overline{\Gamma}^{-1} \left(\int_{\mathcal{M}_{jt}} e^{\sigma(\kappa-1)\xi_{i(m)mj}} \widehat{p}(a(m))^{1-\kappa} \mathrm{d}m \right)^{\frac{1}{1-\kappa}}, \tag{A.5.1}$$

where recall that $\mathcal{M}_{jt} \equiv \{m : A_{mjt} \neq \emptyset\} \subseteq [0, \boldsymbol{M}_t]$ is the subset of industries for which consumer j is aware of at least one firm at time t.

While the price in equilibrium is only a function of age, to compute the integral in (A.5.1) we need to take into account the idiosyncratic sets $\mathcal{M}_{jt}$ and the consumer sorting based on preferences, $\xi_{i(m)mj}$. As it turns out, our assumptions allow us to conveniently simplify this computation.

On the one hand, given that firm demand and prices are only a function of the sizes of non-empty awareness sets, we can replace $\mathcal{M}_{jt}$ with an integral over the unnormalized age distribution, $\boldsymbol{M}_t\Phi_t(a)$, weighted by the proportion of firms that consumers are aware of, $1 - f_0(a)$. On the other hand, the idiosyncratic match value $\xi_{i(m)mj}$ can be shown to be a function of industry age a in expectation. To see this, recall that in the symmetric price equilibrium the consumer simply chooses the product with the highest match value. That is, for a given awareness set $A \neq \emptyset$ of size $\widehat{n} = |A|$, and given preferences $\vec{\xi}(A) \equiv [\xi_1, \ldots, \xi_{\widehat{n}}]^\top \in \mathbb{R}^{\widehat{n}}$ over the firms in this set, the consumer chooses product $i(A) \equiv \{i \in A : \xi_i > \xi_{i'}, \forall i' \in A \backslash \{i\}\}$. Denote the distribution of the maximum $\xi \equiv \max\{\xi_i : i \in A\}$ draw as a function of the awareness set size as the cdf $G_{(\widehat{n})}(\xi)$, with corresponding pdf $g_{(\widehat{n})}(\xi)$. Then, moving the $\overline{\Gamma}^{-1}$ term inside the expression and using equation (18), we can write equation (A.5.1) as:

$$\boldsymbol{P}_t = \left(\boldsymbol{M}_t \int_0^{+\infty} (1 - f_0(a)) \widehat{p}(a)^{1-\kappa} \mathbb{E}_a \left[\Xi(a) \right] \phi_t(a) \mathrm{d}a \right)^{\frac{1}{1-\kappa}}, \tag{A.5.2}$$

where

$$\Xi(a) \equiv \overline{\Gamma}^{\kappa-1} \int_{-\infty}^{+\infty} e^{\sigma(\kappa-1)\xi} g_{(\widehat{n})}(\xi) \mathrm{d}\xi. \tag{A.5.3}$$

Notice that these transformations already imply that the price index is common across consumers, $P_{jt} = \boldsymbol{P}_t$. Next, the order statistic $G_{(\widehat{n})}(\xi)$ is computed as the distribution of the maximum of $\widehat{n}$ draws from the $G(\xi; \mu)$ re-centered Gumbel distribution introduced in equations (A.2.4a)-(A.2.4b). Therefore, by independence of preferences (Assumption 1), $G_{(\widehat{n})}(\xi)$ is determined by the product of the Gumbel cdf's, or:

$$G_{(\widehat{n})}(\xi) = \prod_{h=1}^{\widehat{n}} G(\xi; \mu_h) = e^{-\mu e^{-\xi}} \left(e^{-\mu_{-i} e^{-\xi}} \right)^{\widehat{n}-1} = e^{-(\mu + (\widehat{n}-1)\mu_{-i}) e^{-\xi}}, \tag{A.5.4}$$

where μ denotes the firm's current level of targeting, and we have used (A.2.4a)-(A.2.4b) and Assumption 4 in the second equality. Differentiating to find the pdf:

$$g_{(\widehat{n})}(\xi) = (\mu + (\widehat{n}-1)\mu_{-i}) e^{-\xi} e^{-(\mu + (\widehat{n}-1)\mu_{-i}) e^{-\xi}}. \tag{A.5.5}$$

This allows us to write:

$$\begin{aligned}
\Xi(a) &= \overline{\Gamma}^{\kappa-1} \int_{-\infty}^{+\infty} e^{\sigma(\kappa-1)\xi} g_{(\widehat{n})}(\xi) \mathrm{d}\xi \\
&= (\mu + (\widehat{n}-1)\mu_{-i}) \overline{\Gamma}^{\kappa-1} \int_{-\infty}^{+\infty} e^{-(1-\sigma(\kappa-1))\xi} e^{-(\mu + (\widehat{n}-1)\mu_{-i}) e^{-\xi}} \mathrm{d}\xi \\
&= (\mu + (\widehat{n}-1)\mu_{-i}) \overline{\Gamma}^{\kappa-1} (\mu + (\widehat{n}-1)\mu_{-i})^{-(1-\sigma(\kappa-1))} \Gamma\big(1 - \sigma(\kappa-1)\big) \\
&= (\mu + (\widehat{n}-1)\mu_{-i})^{\sigma(\kappa-1)},
\end{aligned} \tag{A.5.6}$$

where, to go from the second to the third line, we have used the fact that, for any two numbers $a_1, a_2 > 0$, $\int_{-\infty}^{+\infty} e^{-a_1 x} e^{-a_2 e^{-x}} \mathrm{d}x = a_2^{-a_1} \Gamma(a_1)$, where $\Gamma(.)$ is the Gamma function, and recall that $\overline{\Gamma} \equiv \Gamma\big(1 - \sigma(\kappa-1)\big)^{\frac{1}{1-\kappa}}$. Plugging this result back into (A.5.2):

$$\boldsymbol{P}_t = \left(\boldsymbol{M}_t \int_0^{+\infty} (1 - f_0(a)) \widehat{p}(a)^{1-\kappa} \mu^{\sigma(\kappa-1)} \mathbb{E}_a \left[\left(1 + (\widehat{n}-1) \frac{\mu_{-i}}{\mu} \right)^{\sigma(\kappa-1)} \right] \phi_t(a) \mathrm{d}a \right)^{\frac{1}{1-\kappa}}. \tag{A.5.7}$$

Finally, in a symmetric pricing equilibrium, it must be that $\mu = \mu_{-i} = \mu(a) = \mu_0 e^{1-s(a)}$, and the price index (A.5.7) becomes:

$$\boldsymbol{P}_t = \left(\boldsymbol{M}_t \int_0^{+\infty} (1 - f_0(a)) \widehat{p}(a)^{1-\kappa} \mu(a)^{\sigma(\kappa-1)} q(a) \phi_t(a) \mathrm{d}a \right)^{\frac{1}{1-\kappa}}, \tag{A.5.8}$$

where $q(a) \equiv \mathbb{E}_a\left[\widehat{n}^{\sigma(\kappa-1)}\right]$ was defined in equation (21). To compute real income, we use the same arguments that allowed us to write equation (A.5.2) from equation (A.5.1) in order to write the composite good in equation (12) as follows:

$$Y_{jt} = \overline{\Gamma} \left(\int_{\mathcal{M}_{jt}} \left(e^{\sigma \xi_{i(m)mj}} y_{i(m)}(a(m)) \right)^{\frac{\kappa-1}{\kappa}} \mathrm{d}m \right)^{\frac{\kappa}{\kappa-1}}, \tag{A.5.9}$$

where, once again, $a(m)$ is the age of industry m, and $i(m)$ is the product that the consumer purchases in this industry. Developing equation (A.5.9) gives:

$$Y_{jt} = \Omega_{jt}\overline{\Gamma}^{\kappa}\left(\int_{\mathcal{M}_{jt}} e^{\sigma(\kappa-1)\xi_{i(m)mj}} p(a(m))^{1-\kappa}\mathrm{d}m\right)^{\frac{\kappa}{\kappa-1}} \tag{A.5.10}$$

$$= \Omega_{jt}\overline{\Gamma}^{\kappa}\boldsymbol{P}_t^{\kappa}\left(\int_{\mathcal{M}_{jt}} e^{\sigma(\kappa-1)\xi_{i(m)mj}} \widehat{p}(a(m))^{1-\kappa}\mathrm{d}m\right)^{\frac{\kappa}{\kappa-1}} = \Omega_{jt}, \tag{A.5.11}$$

where in the first equality we have used equation (15), in the second equality we have used $p(a(m)) = \widehat{p}(a(m))/\boldsymbol{P}_t$, and in the third equality we have used (A.5.1) to simplify all the terms. This shows that aggregate real income Ω_{jt} equals total output from the composite good Y_{jt}. To express total output Y_{jt} as a function of aggregate capital and TFP (implying $Y_{jt} = \boldsymbol{Y}_t$), recall that aggregate labor demand is given by equation (34):

$$1 = \boldsymbol{L}_t \equiv \boldsymbol{M}_t \int_0^{+\infty} L_t(a)\phi_t(a)\mathrm{d}a, \tag{A.5.12}$$

where $L_t(a) = Nl_t(a)$ is the industry's labor demand, equal to:

$$L_t(a) = (1-\alpha)(1-f_0(a))\mu(a)^{\sigma(\kappa-1)}\boldsymbol{mc}_t^{1-\kappa}\Lambda(a)^{-\kappa}q(a)\frac{\boldsymbol{Y}_t}{w_t} \tag{A.5.13}$$

by equations (20) and (28), and using that $\boldsymbol{\Omega}_t = \boldsymbol{Y}_t$. Next, divide both sides of equation (A.5.8) by $\boldsymbol{P}_t$, use $\widehat{p}(a) = p(a)\boldsymbol{P}_t$ and $p(a) = \Lambda(a)\boldsymbol{mc}_t$, and solve for $\boldsymbol{mc}_t$ to find:

$$\boldsymbol{mc}_t = \boldsymbol{M}_t^{\frac{1}{\kappa-1}}\boldsymbol{Q}_t, \tag{A.5.14}$$

where $\boldsymbol{Q}_t$ is defined in equation (35). Using (A.5.14) in (A.5.12) and (A.5.13):

$$w_t\boldsymbol{L}_t = (1-\alpha)\boldsymbol{Q}_t^{1-\kappa}\boldsymbol{Y}_t\int_0^{+\infty}(1-f_0(a))\Lambda(a)^{-\kappa}\mu(a)^{\sigma(\kappa-1)}q(a)\phi_t(a)\mathrm{d}a \tag{A.5.15}$$

$$= (1-\alpha)\boldsymbol{Q}_t^{1-\kappa}\boldsymbol{Y}_t\boldsymbol{B}_t\boldsymbol{Q}_t^{\kappa-1} \tag{A.5.16}$$

$$= (1-\alpha)\boldsymbol{B}_t\boldsymbol{Y}_t, \tag{A.5.17}$$

where $\boldsymbol{B}_t$ is defined as in equation (36). On the other hand, recall by equation (A.4.9) that $w_t = (1-\alpha)z\boldsymbol{mc}_t\widetilde{k}_t^{\alpha}$, where $\widetilde{k}_t$ is the capital-labor ratio of an industry. By equation (A.4.8), the capital-labor ratio depends only on input prices and is, therefore, constant across industries, which means that $\boldsymbol{K}_t = \widetilde{k}_t$ (as $\boldsymbol{L}_t = 1$). Therefore:

$$w_t = (1-\alpha)z\boldsymbol{M}_t^{\frac{1}{\kappa-1}}\boldsymbol{Q}_t\boldsymbol{K}_t^{\alpha}. \tag{A.5.18}$$

Putting (A.5.17) and (A.5.18) together, we get:

$$\boldsymbol{Y}_t = \boldsymbol{Z}_t\boldsymbol{K}_t^{\alpha}\boldsymbol{L}_t^{1-\alpha}, \tag{A.5.19}$$

with $\boldsymbol{Z}_t \equiv z\boldsymbol{M}_t^{\frac{1}{\kappa-1}}\boldsymbol{Q}_t\boldsymbol{B}_t^{-1}$. This proves part 1 of the proposition.[33] To prove part 2, we must show that the labor, capital and profit shares of total income $\boldsymbol{Y}_t$ are given by $(1-\alpha)\boldsymbol{B}_t$, $\alpha\boldsymbol{B}_t$, and $1-\boldsymbol{B}_t$, respectively. In (A.5.17) we already obtained that $\frac{w_t\boldsymbol{L}_t}{\boldsymbol{Y}_t} = (1-\alpha)\boldsymbol{B}_t$. From (A.4.8), recall $\boldsymbol{K}_t = \frac{\alpha}{1-\alpha}\frac{w_t}{r_t+\delta_K}$, from which it follows that $\frac{(r_t+\delta_K)\boldsymbol{K}_t}{\boldsymbol{Y}_t} = \alpha\boldsymbol{B}_t$. Finally, use equations (30), (32), and (A.5.14) to write firm profits as:

$$\pi(a) = (1-f_0(a))\boldsymbol{M}_t^{-1}\boldsymbol{Q}_t^{1-\kappa}\mu(a)^{\sigma(\kappa-1)}q(a)(\Lambda(a)-1)\Lambda(a)^{-\kappa}\frac{\boldsymbol{Y}_t}{N}. \tag{A.5.20}$$

Therefore, using $\boldsymbol{\Pi}_t \equiv \boldsymbol{M}_t\int_0^{+\infty} N\pi(a)\phi_t(a)\mathrm{d}a$, we get:

$$\frac{\boldsymbol{\Pi}_t}{\boldsymbol{Y}_t} = \boldsymbol{Q}_t^{1-\kappa}\int_0^{+\infty}(1-f_0(a))\mu(a)^{\sigma(\kappa-1)}q(a)(\Lambda(a)-1)\Lambda(a)^{-\kappa}\phi_t(a)\mathrm{d}a = 1-\boldsymbol{B}_t, \tag{A.5.21}$$

where the second equality uses the definitions of $\boldsymbol{Q}_t$ and $\boldsymbol{B}_t$ from (35) and (36). In sum, we have found that real income pays for labor, capital and profit income, so that $\boldsymbol{Y}_t = w_t\boldsymbol{L}_t + (r_t+\delta_K)\boldsymbol{K}_t + \boldsymbol{\Pi}_t$, as we wanted to show.

□

A.6 Proof of Proposition 6

Proof. The current-value Hamiltonian of the representative household is given by:

$$\mathscr{H}_t = \frac{\boldsymbol{C}_t^{1-\gamma}}{1-\gamma} + \varrho_t^A\left(r_t\boldsymbol{A}_t + w_t + (r_t+\delta_K)\boldsymbol{K}_t - \boldsymbol{C}_t - \boldsymbol{I}_t^K - \boldsymbol{I}_t^M + z_M\boldsymbol{I}_t^M\boldsymbol{V}_t^0\right) + \varrho_t^K\left(\boldsymbol{I}_t^K - \delta_K\boldsymbol{K}_t\right) + \varrho_t^M\boldsymbol{I}_t^M, \tag{A.6.1}$$

where $\boldsymbol{C}_t$, $\boldsymbol{I}_t^M$ and $\boldsymbol{I}_t^K$ are the control variables, $\boldsymbol{A}_t$ and $\boldsymbol{K}_t$ are the states variables, and $\varrho_t^A, \varrho_t^K, \varrho_t^M \geq 0$ are the multipliers. The complementary slackness condition $\varrho_t^M\boldsymbol{I}_t^M \geq 0$ must hold for all $t \in \mathbb{R}_+$. The sufficient conditions for optimality are:

$$\partial_{\boldsymbol{C}}\mathscr{H}_t = 0 \quad\Leftrightarrow\quad \boldsymbol{C}_t^{-\gamma} = \varrho_t^A, \tag{A.6.2a}$$
$$\partial_{\boldsymbol{I}^M}\mathscr{H}_t = 0 \quad\Leftrightarrow\quad \varrho_t^M = \varrho_t^A(1-z_M\boldsymbol{V}_t^0), \tag{A.6.2b}$$
$$\partial_{\boldsymbol{I}^K}\mathscr{H}_t = 0 \quad\Leftrightarrow\quad \varrho_t^A = \varrho_t^K, \tag{A.6.2c}$$
$$\partial_{\boldsymbol{A}}\mathscr{H}_t = \rho\varrho_t^A - \partial_t\varrho_t^A \quad\Leftrightarrow\quad \partial_t\varrho_t^A = -(r_t-\rho)\varrho_t^A, \tag{A.6.2d}$$
$$\partial_{\boldsymbol{K}}\mathscr{H}_t = \rho\varrho_t^K - \partial_t\varrho_t^K \quad\Leftrightarrow\quad \varrho_t^A(r_t+\delta_K) - \varrho_t^K\delta_K = \rho\varrho_t^K - \partial_t\varrho_t^K. \tag{A.6.2e}$$

From (A.6.2a), $\frac{\partial_t\varrho_t^A}{\varrho_t^A} = -\gamma\frac{\partial_t\boldsymbol{C}_t}{\boldsymbol{C}_t}$, and using (A.6.2d), we obtain the Euler equation:

[33] Notice, moreover, that if we substitute equation (19) at $p = p_{-i} = p$ and $\mu = \mu_{-i} = \mu(a)$ into (A.5.7) and solve for $\boldsymbol{P}_t\boldsymbol{Y}_t$, we obtain $\boldsymbol{P}_t\boldsymbol{Y}_t = \boldsymbol{M}_t\int_0^{+\infty} N\widehat{p}(a)y(a)\phi_t(a)\mathrm{d}a$. In words, aggregate nominal income is fully exhausted by total nominal expenditures in consumption purchases.

$$\frac{\partial_t \boldsymbol{C}_t}{\boldsymbol{C}_t} = \frac{r_t - \rho}{\gamma}. \tag{A.6.3}$$

Notice that, as per condition (A.6.2c), we would have reached the same conclusion if had used (A.6.2e) instead. Using the industry free entry condition, $z_M \boldsymbol{V}_t^0 = 1$ if $\boldsymbol{I}_t^M > 0$, so by (A.6.2a) we obtain $\varrho_t^M = 0$.

□

A.7 Proof of Proposition 7

Proof. Recall that the law of motion for the measure of product categories is:

$$\frac{\partial_t \boldsymbol{M}_t}{\boldsymbol{M}_t} + \delta_M = \frac{z_M \boldsymbol{I}_t^M}{\boldsymbol{M}_t}. \tag{A.7.1}$$

In turn, the law of motion for the age distribution is given by:

$$\partial_t \widehat{\Phi}_t(a) = -\partial_a \widehat{\Phi}_t(a) - \delta_M \widehat{\Phi}_t(a) + z_M \boldsymbol{I}_t^M, \tag{A.7.2}$$

where $\widehat{\Phi}_t(a) \equiv \boldsymbol{M}_t \Phi_t(a)$. Computing the derivatives $\partial_t \widehat{\Phi}_t(a)$ and $\partial_a \widehat{\Phi}_t(a)$ yields:

$$\partial_t \widehat{\Phi}_t(a) = \boldsymbol{M}_t \partial_t \Phi_t(a) + \Phi_t(a) \partial_t \boldsymbol{M}_t \tag{A.7.3}$$

$$\partial_a \widehat{\Phi}_t(a) = \boldsymbol{M}_t \partial_a \Phi_t(a) \tag{A.7.4}$$

Dividing (A.7.2) by $\boldsymbol{M}_t$, and using (A.7.1), (A.7.3) and (A.7.4) gives:

$$\partial_t \Phi_t(a) = -\partial_a \Phi_t(a) + \left(\delta_M + \frac{\partial_t \boldsymbol{M}_t}{\boldsymbol{M}_t} \right) \big(1 - \Phi_t(a)\big). \tag{A.7.5}$$

In a stationary equilibrium, $\partial_t \Phi_t(a) = 0$, $\forall a \geq 0$, and $\partial_t \boldsymbol{M}_t = 0$, so that $\Phi_t(a) = \Phi(a)$ and $\boldsymbol{M}_t = \boldsymbol{M}$. Imposing this on (A.7.5) gives:

$$\partial_a \Phi(a) - \delta_M \big(1 - \Phi(a)\big) = 0. \tag{A.7.6}$$

This is a first-order ordinary differential equation with boundary conditions $\Phi(0) = 0$ and $\lim_{a \to +\infty} \Phi(a) = 1$, which can be solved with simple methods. The solution is:

$$\Phi(a) = 1 - e^{-\delta_M a}. \tag{A.7.7}$$

The corresponding pdf is $\phi(a) = \partial_a \Phi(a) = \delta_M e^{-\delta_M a}$.

□